Flatness of the Hodge filtration on the Gauss – Manin connection implies finite monodromy.

$T^F : \pi_1(X, x_0) \to GL_1(\mathbb{C})$

$\gamma \mapsto T_\gamma^F$

monodromy rep.

Setup

$$\begin{tikzcd} X \setminus D = U \arrow[r, hook, "j"] \arrow[dr, "\pi"'] & X \arrow[d, "f\ \text{proj sm}"] & D \arrow[l, hook'] \ \text{SNCD} \\ & S \ \text{connected sm.} \arrow[d] & \\ & T = \operatorname{Spec} \mathbb{C} & \end{tikzcd}$$

~~diagram holds for a if we take analification.~~

Analyfication

X affine scheme $(X^{an}, \mathcal{O}_{X^{an}})$.

- $|X^{an}| = |X(\mathbb{C})|$
- equip X^{an} with the Euclidean top.
- $\mathcal{O}_{X^{an}}$ sheaf of holomorphic functions.

$U \simeq V \subseteq \mathbb{C}^n$

$U \subset X(\mathbb{C})$

zariski

Zarisk

induced Euclidean top

zariski non closed point.

Rmk: the above setup also holds after taking analyfication.

~~For any $\mathcal{F}^{an} = \mathcal{F}$~~

For any $\mathcal{F}$ on X, define $\mathcal{F}^{an} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{an}}$.

Goal: Construct $(R^n \pi_*^{an} (\underline{\mathbb{Z}}_{U^{an}}), W, F)$ as a polarizable family of mixed Hodge str.

Construction

$$\begin{tikzcd} \underline{\mathbb{Z}}_{U^{an}} & U^{an} \arrow[r, hook] \arrow[dr, "j^{an}"'] & X^{an} \arrow[d, "f^{an}"] \\ & & S^{an} \end{tikzcd}$$

For $n \geq 0$, $R^n j_*^{an}(\underline{\mathbb{Z}})$ on S^{an} is a local sys of ~~Z~~ $\mathbb{Z}$-module of f.t.

$$R^n j_*^{an}(\underline{\mathbb{Q}}) = R^n j_*^{an}(\underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

rational local sys

is the abutment of the Leray spectral sequence in the local systems on S^{an}

$p+q=n$. $E_2^{p,q} = R^p f_*^{an}(R^q j_*^{an}(\underline{\mathbb{Q}})) \Rightarrow R^{p,q} j_*^{an}(\underline{\mathbb{Q}})$

$\Rightarrow$ it defines a decreasing filtration N^i of $R^n j_*^{an}(\underline{\mathbb{Q}})$:

$$N^i = \mathrm{Im}(H^n(S, \tau^{\geq k} R j_* \underline{\mathbb{Q}}) \to H^n(S, R j_* \underline{\mathbb{Q}})).$$

$\tau^{\geq k} R j_* \underline{\mathbb{Q}} : 0 \to 0 \to \cdots \to 0 \to \underset{k\text{th}}{R j_* \underline{\mathbb{Q}}} \xrightarrow{d_{k+1}} R j_* \underline{\mathbb{Q}} \to \cdots$

$R j_* \underline{\mathbb{Q}} : R j_* \underline{\mathbb{Q}} \to R j_* \underline{\mathbb{Q}} \to \cdots$

~~define W_i~~ $N^k = \begin{cases} H^n(S, \underline{\mathbb{Q}}) & k \leq n \\ 0 & k > n \end{cases}$

$$N^0 \supset N^1 \supset N^2 \supset \cdots$$

define $W_i R^n j_*^{an}(\underline{\mathbb{Z}})$ = the inverse image of $N^{2n-i} R^n j_*^{an}(\underline{\mathbb{Q}})$ under (the canonical map) $R^n j_*^{an} \underline{\mathbb{Z}} \to R^n j_*^{an} \underline{\mathbb{Q}}$

$$\begin{array}{ccccc} R^n j_*^{an} \underline{\mathbb{Q}} & \supset N^k & \supset N^{k+1} & \supset \cdots \\ \uparrow & & & \\ R^n j_*^{an} \underline{\mathbb{Z}} & \supset W^{2n-k} & \supset W^{2n-k-1} & \supset \cdots \end{array}$$

$W_\bullet$ increasing. $W_i = 0$ if $i < n$ and W_i = all if $i \geq 2n$.

Γ Hypercohomology :

X top space, $(\mathcal{F}^\bullet, d)$ complex of sheaves on X

$\mathbb{H}^k(X, \mathcal{F}^\bullet) := R^k\Gamma(\mathcal{F}^\bullet, d)$ $\quad \mathcal{F}^\bullet : \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$

if $\mathcal{F}^\bullet$ is $\mathcal{F}^0 \to 0 \to 0 \to \cdots$ ~~then $\mathbb{H}^k(X,\mathcal{F}^\bullet)$~~ =

then $\mathbb{H}^k(X, \mathcal{F}^\bullet) = H^k(X, \mathcal{F}^0)$.

$\mathbb{H}^0(X, \mathcal{F}^\bullet) \simeq \Gamma(\mathrm{Ker}(\mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1))$.

$(\mathcal{F}^\bullet, d)$ $\quad E_r^{p,q}$ spectral sequence

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q} \cap Z_r^{p,q}}$$ where $\mathcal{F}^\bullet$ has biregular decreasing filtr

i.e. ~~$r \gg 0$~~, fix p, q, $r \gg 0$, $Z_r^{p,q}$, $B_r^{p,q}$ stablize

$$E_r = E_\infty = \frac{F^p\mathbb{H}^n(\mathcal{F}^\bullet)}{F^{p+1}\mathbb{H}^n(\mathcal{F}^\bullet)} \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet).$$

$F^p\mathbb{H}^n(\mathcal{F}^\bullet) = \mathrm{Im}(\mathbb{H}^n(X, \mathcal{F}^{\geq p}) \to \mathbb{H}^n(X, \mathcal{F}^\bullet))$.

this filtration is intrinsic for the spectral seq.

the locally free coherent sheaf on S^{an}

$$(*) \quad R^n f_*^{an}(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}} = R^n f_*^{an}((\Omega^\bullet_{X/S}(\log D))^{an})$$

derived functor of Γ of complex of sheaves

$$\simeq R^n f_*(\Omega^\bullet_{X/S}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}}$$

is the abutment of the Hodge ⇒ De Rham spectral seq.

$$\begin{array}{ccc} E_1^{p,q} = R^q f_*^{an}((\Omega^p_{X/S}(\log D))^{an}) & \Rightarrow & R^{p+q} f_*^{an}((\Omega^\bullet_{X/S}(\log D))^{an}) \\ \| & & \| \\ R^q f_*^{an}(\Omega^p_{X/S}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} & & R^{p+q} f_*(\Omega^\bullet_{X/S}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} \end{array}$$

which has E_1 locally free (and degenerate at E_1).

~~⊛ because~~.

Sketch of the proof of ⊛.

U complex manifold $n = \dim U$.

- holomorphic DR complex

$\Omega^\bullet_U \quad 0 \to \mathcal{O}_U \xrightarrow{d} \Omega^1_U \to \cdots \xrightarrow{d} \Omega^n_U \to 0$

$i' : \underline{\mathbb{C}}_U \hookrightarrow \mathcal{O}_U$

$\underline{\mathbb{C}}_U \hookrightarrow \Omega^\bullet_U$ is a ~~resolution of~~ $\underline{\mathbb{C}}_U$.

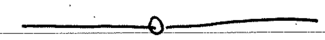
i.e. ~~isomorphic in the derived cat.~~

then $\underline{\mathbb{C}}_U$ and $\Omega^\bullet_U$ are quasi-isomorph

i.e. isomorph in the derived cat.

- $j : U \hookrightarrow X$

$j_* \Omega^\bullet_U \hookleftarrow \Omega^\bullet_X(\log D)$ quasi-isomorph

Example: $U = \operatorname{Spec} \mathbb{C}[t, \frac{1}{t}]$ 

$\downarrow$

$X = \operatorname{Spec} \mathbb{C}[t]$ 

$$\begin{tikzcd}
\Omega^\bullet_U(U) = & \mathbb{C}[t, \frac{1}{t}] \arrow[r, "d"] & \mathbb{C}[t, \frac{1}{t}]\,dt \arrow[r] & 0 \\
\Omega^\bullet_X(\log D) = & \mathbb{C}[t] \arrow[u] \arrow[r, "d"] & \mathbb{C}[t]\,\frac{dt}{t} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

$H^0 = \mathbb{C} \quad H^1 = \mathbb{C}\, d\log t$ for both $\Omega^\bullet_U$, $\Omega^\bullet_X(\log D)$.

so $\quad H^n(U, \underline{\mathbb{C}}) = \mathbb{H}^n(U, \Omega^\bullet_U)$

$\mathbb{H}^n(U, \Omega^\bullet_U) \cancel{\mathbb{H}} = \mathbb{H}^n(X, \Omega^\bullet_X(\log D))$.

define $F_s^i = \mathrm{Im}\left(\mathbb{H}^n(S^{an}, \tau^{\geq i} R\Psi_0^{an}(\mathbb{C}))_s \longrightarrow \mathbb{H}^n(S^{an}, R\Psi_0^{an}(\mathbb{C}))_s \right)$

s.t. $$F_s^i(gr_n^W) = \frac{(W_n \otimes \mathbb{C})_s \cap F_s^i}{(W_{n-1} \otimes \mathbb{C})_s \cap F_s^i}$$

Rmk: two filtration $W_\bullet$, $F^\bullet$ exactly comes from the two spectral seq.

Prop (Deligne – Hodge). The triple $(R^n \pi_*^{an}(\mathbb{Z}), W, F)$ defined ~~at~~ above is a polarizable family of mixed Hodge structure on S^{an}.

Sketch of the proof:

- a family of mixed Hodge structure. (follows from the Deligne's theorem)
- it's polarizable.

$$\begin{tikzcd} D \arrow[r, hook, "i"] & X \arrow[d, "f"] & U \arrow[l, hook', "j"'] \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

consider the Leray spectral sequence

$$E_2^{p,q} = R^p f_*^{an}(R^q j_*^{an}\mathbb{Z}) \Rightarrow R^{p+q}\pi_*^{an}(\mathbb{Z})$$

and
$$E_2^{p,q} = \begin{cases} \bigoplus\limits_{i_1<\cdots<i_q} R^p f^{an}(\phi\, i_1 \dots i_q)_*(\mathbb{Z})(-q) & \text{if } q \neq 0 \\ R^p f_*^{an}(\mathbb{Z}) & \text{if } q = 0 \end{cases}$$

where $\cdot(-q) := \cdot \otimes_{\mathbb{Z}} \mathbb{Z}(-q)$,
$$\begin{tikzcd} D_{i_1}\cap\cdots\cap D_{i_n} \arrow[r, hook] \arrow[drr, "f(i_1\dots i_n)"'] & D \arrow[r, hook] & X \arrow[d, "f"] \\ & & S \end{tikzcd}$$

~~where~~

~~because direct sum of polarizable family are polarizable, and Tate twist $H(-q)$ of polarizable family~~

because direct sum of polarizable family are polarizable
Tate twist $H(-q)$ ...

~~App~~

Apply the following proposition to X and to all intersections $D_{i_1}\cap\cdots\cap D_{i_q}$.

Prop: (Hodge's index Theorem). Let $f: X \to S$ be a projective and smooth morphism of $\mathbb{C}$-schemes. Then for any ~~integer intersection~~ integer $n \geq 0$, $R^n f_*^{an} \mathbb{Z}$ is polarizable.

Sketch of the proof:

assume that X/S has geometrically closed fibers, and is of constant relative dim N.

Let $L \in H^2(X^{an}, \mathbb{Z})$ be (the cohomology class of a hyperplane section, i.e.) the inverse image under $pr_2 \circ i$.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & S \times \mathbb{P}_{\mathbb{C}} \arrow[r, "pr_2"] \arrow[d, "pr_1"] & \mathbb{P}_{\mathbb{C}} \\ & S & \end{tikzcd}$$

of the class of a hyperplane in $H^2(\mathbb{P}_{\mathbb{C}}^{an}, \mathbb{Z})$

By the "Hard Lefschetz thm" the interated ~~cup~~ cup-product with L

$$L^i : R^{N-i} f_*^{an}(\mathbb{Z})(-i) \longrightarrow R^{N+i} f_*^{an}(\mathbb{Z})$$

is an isogeny. (i.e. becomes an isomorphism when $\otimes \mathbb{Q}$).

Thus sufficient to show that $R^n f_*^{an}(\mathbb{Z})$ is polarizable for $n \leq N$.

True by arguing with the "primitive decomposition" and "Hodge index theorem".

Rmk

$$\begin{tikzcd} U^{an} \arrow[r, hook, "j^{an}"] \arrow[dr, "\tau^{an}"'] & X^{an} \arrow[d, "f^{an}"] \\ & S^{an} \end{tikzcd}$$

i) When $S = \mathrm{Spec}\,\mathbb{C}$, the mixed Hodge str on $H^n(U^{an}, \mathbb{Z})$ $\left(= H^n(S^{an}, \tau^{an}_* \mathbb{Z})\right)$ depends only on U, not not the compacitificaton

$\underbrace{U \hookrightarrow X}_{X/\mathbb{C}}$ ~~choose~~ proper smooth variety s.t. $D = X \setminus U$ is a SNCD.

ii) The mixed Hodge str is functorial in U, i.e. if $h : U \to V$ is a morph of smooth $\mathbb{C}$-schs. the induced morph $h^* : H^n(V^{an}, \mathbb{Z}) \to H^n(U^{an}, \mathbb{Z})$ are morph of mixed Hodge str.

~~Follow the~~ above (TFAE).

Prop: ~~Hypotheses~~ Hypotheses as in (4.3.0), the following conditions are equivalent, for any integer $n \geq 0$.

0.) The Hodge filtration F on the locally free sheaf $R^n f_*(\Omega^{\bullet}_{X/S}(\log D))$ on S is horizontal for the ~~(GM)~~ Gauss–Manin connection ~~GM~~ ∇. ~~$\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_S$~~ flat ~~$\nabla^2 = 0$~~

1.) The Hodge filtration F of the family of mixed Hodge structure on S^{an}, $(R^n \tau_*^{an} \mathbb{Z}_{U^{an}}, W, F)$, is locally constant (i.e. it comes from a filtration of $R^n \tau_*^{an}(\mathbb{C})$ by sub-local systems).

2.) There exists a finite étale covering $\varphi: S' \to S$ s.t. $(\varphi^{an})^*(R^n \tau_*^{an}(\mathbb{Z}), W, F)$ is a constant family of mixed Hodge str on $(S')^{an}$

f.p. flat, unramified

3.) There exists a finite étale covering $\varphi: S' \to S$ s.t. $\varphi^*(R^n f_*(\Omega^{\bullet}_{X/S}(\log D)), \nabla)$ is isomorphic to $((\mathcal{O}_{S'})^{b_n}, d)$ as a coherent ~~$\mathcal{O}$~~ $\mathcal{O}_S$-module with connection ~~(b_n = rank of~~ (b_n = rank of $R^n f_*(\Omega^{\bullet}_{X/S}(\log D))$).

Rmk: • By (4.~~X~~.6), $R^n \tau_*^{an}(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S^{an}} = R^n f_*^{an}((\Omega^{\bullet}_{X/S}(\log D))^{an})$
~~we have~~ $0) \Rightarrow 1)$. $= R^n f_*(\Omega^{\bullet}_{X/S}(\log D)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}}$.

• By (4.2.2.3), Prop 1), and Riemann's ~~exten~~ existence (4.2.2.3) theorem, we have ~~(4.3.3.1) ⇒ (4.3.3.2)~~ $1) \Rightarrow 2)$.

Prop: Let S be a topological space ~~and~~ as in (4.2.1.3). Let $(H_{\mathbb{Z}}, F, W)$ be a family of mixed Hodge structures on S, s.t. each of the associated graded family $(gr^W_n H_{\mathbb{Z}}, F)$ of pure Hodge structures is polarizable. Suppose that the Hodge filtration F is locally constant, in the sense that it comes from a filtration by sub-local system of ~~complexi~~ ~~complexification~~ complexified local ~~sy~~ system $H_{\mathbb{C}}$.

Then there exists a finite étale covering $\pi : S' \to S$ s.t. the inverse image $\pi^*(H_{\mathbb{Z}}, W, F)$ of $(H_{\mathbb{Z}}, W, F)$ on S' is a constant family of mixed Hodge str.

Riemann's ~~exten~~ existence theorem:

Let S be a smooth connected $\mathbb{C}$-scheme, and let S^{an} denote the corresponding complex ~~man~~ manifold. Denote by $Etale(S)$ (resp $Etale(S')$) the category of finite étale covering of S (resp S^{an}). Then the natural functor $Etale(S) \to Etale(S^{an})$ is an equivalence of categories.

Rmk: 0) ⟹ ~~B~~ 3). first ingredient in the proof of the Grothendieck - Katz conjecture for Gauss - ~~Manin~~ Manin connections.