Unipotent characters

Notation :

let | $G/\mathbb{F}_q$ be a conn. red. gp def over $\mathbb{F}_q$ $\longleftrightarrow \overset{\Psi(G)}{\overset{\shortparallel}{(X, Y, \Phi, \check{\Phi})}}$

| $F : G \to G$ a Steinberg endomorphism.

| T_0 a max split F-stable torus

| $X_{\mathbb{R}} := X \otimes_{\mathbb{Z}} \mathbb{R}$, S = simple ref. of W

Recall :

* for any F-stable max torus T, T^F is conj to $T_0[w] := T_0^{wF}$

* $$\mathrm{Irr}(G^F) = \bigsqcup_{\substack{[(T,\theta)] \\ \in \mathcal{X}(G,F)/\sim_{geom}}} \mathcal{E}_{[(T,\theta)]}$$

$$\| \quad [(T,\theta)] \longleftrightarrow [(w,\theta)] \dashrightarrow [(w^*,s)] \dashrightarrow [(T^*,s)]$$

$$= \bigsqcup_{\substack{[s] \\ ss\ G^*\text{-class} \\ F\text{-stable}}} \overbrace{\bigsqcup_{\substack{[s']\ G^{*F^*}\text{-cls} \\ s \underset{G^*}{\sim} s'}} \mathcal{E}(G^F, s')}$$

$\theta \mapsto s$
gp iso

where | $\mathcal{E}_{[(T,\theta)]} = \{ \rho \in \mathrm{Irr}(G^F) \mid \exists (T', \theta') \sim_{geom} (T,\theta) : \langle R_{T'}^G(\theta'), \rho \rangle \neq 0 \}$

| $\mathcal{E}(G^F, s') = \{ \rho \in \mathrm{Irr}(G^F) \mid \exists T^* \text{ containing } s' : \langle R_{T'}^G(\theta'), \rho \rangle \neq 0 \}$

$\mathcal{E}(G^F,s')$: $s' \in (G^*)^{F^*}$; $(T^*, s') \longleftrightarrow (T', \theta')$

Want to understand unip. char. : combinatorial data, degree

→ invariants

↪ relat° with Irr(W)

I. First approach

Def 1.

$\rho \in \mathrm{Irr}(G^F)$ is called a unipotent character if

$$\exists \text{ max } F\text{-stable torus } T : \langle R_T^G \mathbb{1}_T, \rho \rangle \neq 0$$

We denote this set by $\mathrm{Uch}(G^F)$.

Rmk:

$* \ \forall T, \ (T, \mathbb{1}_T) \underset{geo}{\sim} (T_0, \mathbb{1}_{T_0})$.

$\Rightarrow \mathrm{Uch}(G^F) = \mathcal{E}_{[(T_0, \mathbb{1}_{T_0})]}$ geom. series

$* \ [(T_0, \mathbb{1}_{T_0})] \longleftrightarrow [(w, \mathbb{1}_{T_0})] \longleftrightarrow [(w^*, 1_{T_0^*})]$

$\Rightarrow \mathrm{Uch}(G^F) = \mathcal{E}(G^F, 1)$ rational series.

Example: $0 \to \mathbb{1} \to \mathrm{Ind}_{T_0}^{G} \mathbb{1} \to \mathrm{St} \to 0$

$* \ \mathbb{1}_G = \frac{1}{|W|} \sum_{w \in W} R_w^{\mathbb{1}}$

then $\forall w \in W$, $\langle R_w^{\mathbb{1}}, \mathbb{1}_G \rangle$

$$= \frac{1}{|W|} \sum_{w' \in W} \langle R_w^{\mathbb{1}}, R_{w'}^{\mathbb{1}} \rangle$$

$$= \frac{1}{|W|} \sum_{w' \in W} \# \{ x \in W \mid w'^{x} = w \}$$

$$= \frac{1}{|W|} \sum_{w' \in \mathrm{Orb}(w)} \# \mathrm{Stab}_W(w)$$

$$= 1$$

$$* \; St_G = \frac{1}{|W|} \sum_{w\in W} (-1)^{\ell(w)} R_w^{\mathbb{1}}$$

$$\Rightarrow \forall w \in W, \; \langle R_w^{\mathbb{1}}, St_G \rangle = (-1)^{\ell(w)}$$ since $\ell(sws) = \ell(w) \pm 2$ or $\ell(w)$

Fact:

$$\chi_{reg} = \sum_{\rho \in Irr(G^F)} \rho(1)\rho. = \frac{1}{|W|} \sum_{\substack{(w,\theta) \\ \in \mathcal{X}(W,F)}} R_w^\theta(1) R_w^\theta$$

In part., $\forall \rho \in Uch(G^F)$, $\rho(1) = \frac{1}{|W|} \sum_{w\in W} R_w^{\mathbb{1}}(1) \langle R_w^{\mathbb{1}}, \rho \rangle$

Goal VI: In order to compute $\rho(1)$, we need to know

Def 2.

Let $t \in T$

t is called a regular elem^T if $C_G^\circ(t) = T$.

We denote this set by T_{reg}.

Fact:

$\forall \rho \in Uch(G^F)$, $\forall s \in T_{reg}^F$,

$$\rho(s) = \langle R_T^G \mathbb{1}, \rho \rangle$$

Lemma 3.

$$\exists \, C_{\psi(G)} > 0 : \frac{|T_{reg}^F|}{|T|} \geq 1 - \frac{C_{\psi(G)}}{q}$$

Prop 4.

Assume $\forall T$, $\dfrac{|T_{reg}^F|}{|T^F|} > 1 - 2^{-|W|}$

then $\rho \in \mathrm{Irr}(G^F)$ is unip.

$\iff \forall T$, $\rho_{|T_{reg}^F}$ is constant.

$\hookrightarrow$ require a lot of knowledges about the explicit gp

II. Reduction.

Prop 5:

let H be another conn. red gp with $\tilde{F} : H \to H$

$\pi : G \to H$ surj s.t. $\tilde{F} \circ \pi = \pi \circ F$, $\mathrm{Ker}\,\pi \subset Z(G)$

then $\mathrm{Uch}(H^{\tilde{F}}) \longrightarrow \mathrm{Uch}(G^F)$ is a bij.

$\rho \longmapsto \rho \circ \pi_{|G^F}$

Application: $\pi : G \to G_{ad} := G/Z(G)$, $F_{ad} \circ \pi = \pi \circ F$

thus we can study $\mathrm{Uch}(G_{ad}^{F_{ad}})$ where G_{ad} is

semisimple of adjoint type. (i.e. $X = \mathbb{Z}\Phi$)

but we can do even better.

$$G_{ad} = \prod_{i=1}^{r} G_i \quad \text{with } G_i \text{ ss adj. } F_{ad}\text{-stable } F_{ad}\text{-simple}$$

$$= \prod_{i=1}^{r} \prod_{j=0}^{k_i-1} F_{ad}^{j}(H_i) \quad \text{with } H_i < G_i \text{ simple}$$

we have embeddings:

$$\begin{aligned} \iota_i : H_i &\longrightarrow G_i \\ g &\longmapsto g\, F_{ad}(g) \cdots F_{ad}^{k_i-1}(g) \end{aligned}$$

which induces:

$$\iota_i : H_i^{F_{ad}^{k_i}} \xrightarrow{\sim} G_i^{F_{ad}}$$

thus $\mathrm{Uch}(G_{ad}^{F_{ad}}) \simeq \prod_{i=1}^{r} \mathrm{Uch}(H_i^{F_{ad}^{k_i}})$

i.e. we can reduce to the case where G is simple of adjoint type.

<u>Example</u>:

A $GL_n \rightsquigarrow PGL_n$, B $\mathrm{Spin}_{2n+1} \rightsquigarrow SO_{2n+1}$,

C $Sp_{2n} \rightsquigarrow PSp_{2n}$, D $SO_{2n} \rightsquigarrow PSO_{2n}$

Def 6:

let $\Psi = (X, Y, \Phi, \check{\Phi})$ a Root datum with its Weyl group $W \subset \mathrm{Aut}(X) \subset GL(X_{\mathbb{R}})$

$\varphi_0 \in N_{GL(X_{\mathbb{R}})}(W)$ of finite order and "nice"

$\exists q > 0$: $q\varphi_0$ nice

$\approx$ comes from F.

then $\mathbb{G} := ((X, Y, \Phi, \check{\Phi}), \varphi_0 W)$ is called a *complete root datum*

$$|\mathbb{G}| := q^{|\Phi|}\left(\frac{1}{|W|}\sum_{w\in W} \det(q\,\mathrm{id}_{X_{\mathbb{R}}} - \varphi_0 \circ w^{-1})^{-1}\right)^{-1} \in \mathbb{R}(q)$$

is called *order polynomial* of $\mathbb{G}$.

Example:

$F^d = F_{R_q}$, let $a \in \mathbb{R}_+$ s.t. $a^d = q$

then F induces a map $a\varphi_0 \in GL(X_{\mathbb{R}})$ where $\mathrm{ord}(\varphi_0) < \infty$

$\Rightarrow$ $(\Psi(G), \varphi_0 W)$ is a complete datum.

$|G^F| = |\mathbb{G}|(q)$.

Def 7:

let $(\Psi(G), \varphi_0 W)$ associated to (G, F)

We def the *degree polynomial* of $\rho \in \mathrm{Irr}(G^F)$.

$$\mathbb{D}_\rho := \frac{1}{|W|}\sum_{(w,\theta)\in \mathcal{X}(W,F)} (-1)^{\ell(w)} \langle R_w^\theta, \rho\rangle\, q^{-N} \frac{|\mathbb{G}|}{|T_w|} \in \mathbb{R}[q]$$

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where $\begin{cases} N = |\Phi^+| \\ |\mathbb{T}_w| \text{ is the order polynomial of } T_0[w] \end{cases}$

We def also the invariants : a_p and A_p st :

$$\mathbb{D}_p = \sum_{i=a_p}^{A_p} c_i q^i \quad \text{with } c_i \neq 0$$

invariant bcz later they will be cst on families

Rmk :

φ_0 induces an auto. σ on the Coxeter graph i.e.

$\sigma(S) = S$ (and $\ell = \ell \circ \sigma$)

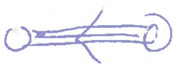
Def 8 :

σ is ordinary if it is an auto on the Dynkin diag.

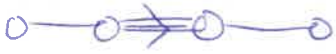
Rmk :

σ not ordinary <=> $\sigma \neq id$ in type B_2

G_2



F_4



Thm 9.

$\exists$ finite set $\bar{X}(W, \sigma)$:

$$\begin{array}{ccc} \mathrm{Uch}(G^F) & \xleftrightarrow{1-1} & \bar{X}(W, \sigma) \\ \rho & \longmapsto & \bar{x}_\rho \end{array}$$

and $$\mathbb{D}_\rho = \frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)} \underbrace{\langle R_w, \rho \rangle}_{\text{want to know this}} \underbrace{\frac{|G|_{q'}}{|\mathbb{T}_w|}}_{\text{generic}}$$

Thm 10:

$\exists$ bijection : $\{ \rho \in \mathrm{Uch}(G^F) \mid \langle \mathrm{Ind}_T^G \mathbb{1}, \rho \rangle \neq 0 \} \longleftrightarrow \mathrm{Irr}(W)$

Rmk.

in type A_n, $\bar{X}(W, \sigma) = \mathrm{Irr}(W)$, otherwise one cuspidal

or ${}^2A_n$ with $n \neq \frac{s^2+s}{2} - 1$ (similar for other types)

$n \neq s^2+s$ for B_n / C_n

Goal V2 : * compute $\langle R_w, \rho \rangle$

* understand $\mathrm{Irr}(W)$

III. Representations of W.

From now, we fix $T := \{ wsw^{-1} \mid w \in W,\ s \in S \}$.

and $\varepsilon : w \in W \longmapsto (-1)^{\ell(w)}$ the sign character in $\mathrm{Irr}(W)$ $\longleftrightarrow \mathrm{St} \in \mathrm{Uch}(G^F)$

Def 1.1.

let $\phi \in \mathrm{Irr}(W)$.

we def : $\omega_\phi := \sum_{t \in T} \frac{\phi(t)}{\phi(1)} \in \mathbb{Z}$

Notation :

for $J \subset S$, $W_J = \langle J \rangle$, $\mathrm{Ind}_I^J := \mathrm{Ind}_{W_I}^{W_J}$.

Def 1.2 :

We def the a-invariant inductively :

- $W = \{1\}$: $a : \mathrm{Irr}(W) \longrightarrow \mathbb{Z}$, $\mathbb{1} \longmapsto 0$

- $W \neq \{1\}$: supp a is def over $\mathrm{Irr}(W_J)$ for any $J \subsetneq S$.

  let $\phi \in \mathrm{Irr}(W)$

$$a_\phi' := \max \left\{ a_\psi \;\middle|\; \begin{array}{l} \psi \in \mathrm{Irr}(W_J),\ J \subsetneq S, \\ \langle \mathrm{Ind}_J^S \psi, \phi \rangle \neq 0 \end{array} \right\}$$

$$a_\phi := \begin{cases} a_\phi' & \text{if } a_{\varepsilon\phi}' - a_\phi' \leq \omega_\phi \\ a_{\varepsilon\phi}' - \omega_\phi & \text{otherwise} \end{cases}$$

Prop 13.

$$* \ 0 \leq a_{\phi'} \leq a_\phi \leq \ell(w_0)$$

$$\left(* \ a_{\varepsilon\phi} - a_\phi = \omega_\phi \right)$$

$$* \ \phi_1 \boxtimes \phi_2 \in \mathrm{Irr}(W_1 \times W_2) \Rightarrow a_{\phi_1 \boxtimes \phi_2} = a_{\phi_1} + a_{\phi_2}$$

Example:

$$a_{1_W} = 0 \quad \text{and} \quad a_\varepsilon = \ell(w_0)$$

Def 14 the A-invariant.

$$A_\phi := \ell(w_0) - a_\phi - \omega_\phi = \ell(w_0) - a_{\varepsilon\phi}$$

Prop 15

$$* \ 0 \leq A_\phi \leq \ell(w_0)$$

$$* \ A_{\phi_1 \boxtimes \phi_2} = A_{\phi_1} + A_{\phi_2}$$

strict

Def 16. $(\mathrm{Irr}(W), \lneq)$ inductively.

$* \ W = \{1\}$: ✓

$* \ W \neq \{1\}$: $\phi \leq \phi'$ if $\exists (\phi_0 = \phi, \ldots, \phi_m = \phi') \in \mathrm{Irr}(W)^{m+1}$:

$\forall 1 \leq i \leq m, \ \exists I_i \subsetneq S, \ \psi_i, \psi_i' \in \mathrm{Irr}(W_{I_i}) : \psi_i \lneq \psi_i'$ and

either $\begin{cases} \langle \phi_{i-1}, \mathrm{Ind}_{I_i}^{S} \psi_i \rangle \neq 0 \\ \langle \phi_i, \mathrm{Ind}_{I_i}^{S} \psi_i' \rangle \neq 0 \\ a_{\phi_i} = a_{\psi_i'} \end{cases}$ or $\begin{cases} \langle \varepsilon\phi_{i-1}, \mathrm{Ind}_{I_i}^{S} \psi_i' \rangle \neq 0 \\ \langle \varepsilon\phi_i, \mathrm{Ind}_{I_i}^{S} \psi_i \rangle \neq 0 \\ a_{\varepsilon\phi_{i-1}} = a_{\psi_i'} \end{cases}$

Def 17.

$$\phi \sim \phi' \overset{def}{\iff} \phi \preceq \phi' \text{ and } \phi' \preceq \phi$$

We call those equiv. classes *families*.

Rmk:

- $\phi \preceq \phi' \Rightarrow \begin{cases} \varepsilon\phi' \preceq \varepsilon\phi \\ a_{\phi'} \leq a_{\phi} \end{cases}$
- a-funct° is constant on a family.

Example:

- $W = S_2$, $\mathrm{Irr}(W) = \{\mathbb{1}_W, \varepsilon\}$.

  $0 \longrightarrow \mathbb{1}_W \longrightarrow \mathrm{Ind}_{\emptyset}^{S} \mathbb{1}_1 \longrightarrow \varepsilon \longrightarrow 0$

  $a_{\mathbb{1}_W} = 0 = a_{\mathbb{1}_1}$ and $a_{\varepsilon} = 1$

  $\Rightarrow \varepsilon \preceq \mathbb{1}$ and $\mathbb{1} \not\preceq \varepsilon$
- all families in $\mathrm{Irr}(S_n)$ are singletons.

Rmk:

$\exists$ some b and B functions:

$$\begin{cases} a_{\phi} \leq b_{\phi} \leq B_{\phi} \leq A_{\phi} \\ \forall \mathcal{F} \in \mathrm{Irr}(W)/\sim,\ \exists!\, \phi \in \mathcal{F}.\ a_{\phi} = b_{\phi} \end{cases}$$

we call it the *special character* in $\mathcal{F}$.

IV. Multiplicities

Idea: want to have something closer to be a true character than DL characters, so that the mult. is easier to compute

Consider $\tilde{W} := W \rtimes \langle \sigma \rangle$

Def 18. let $\phi \in \mathrm{Irr}(W)^{\sigma}$

* a σ-conj. class of $\tilde{w} \in \tilde{W}$ is $\{ w \tilde{w} \sigma(w)^{-1} \}_{w \in W}$

* a σ-extension $\tilde{\phi} \in CF_{\sigma}(\tilde{W})$ is s.t.

$$\forall w \in W, \quad \tilde{\phi}(w) = \mathrm{Tr}(\tau(w) E) \quad \text{where} \begin{cases} \phi = \mathrm{Tr}(\tau) \\ \tau \circ \sigma = E \cdot \tau E^{-1} \end{cases}$$

* the almost character associated to ϕ:

$$R_{\tilde{\phi}} := \frac{1}{|W|} \sum_{w \in W} \tilde{\phi}(w) R_w^{\mathbb{1}}$$

Example:

$R_{\tilde{\mathbb{1}}_W} = \mathbb{1}_G$ and $R_{\tilde{\varepsilon}} := St_G$

Def 19.

We def a graph (V, E) s.t.:

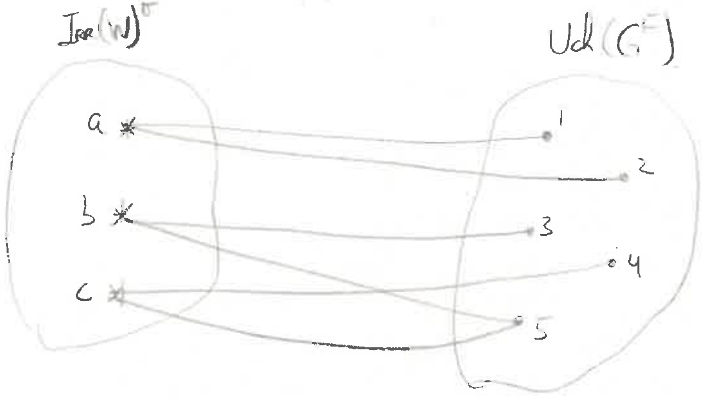
* $V = \mathrm{Uch}(G^F)$

* $\{\rho_1, \rho_2\} \in E \overset{\text{def}}{\iff} \exists \phi \in \mathrm{Irr}(W)^{\sigma} : \forall i = 1, 2, \ \langle \rho_i, R_{\tilde{\phi}} \rangle \neq 0$

A connected component of the graph is called a family

Def 20.

$\phi, \phi' \in \mathrm{Irr}(W)^\sigma$ are equiv if R_ϕ and $R_{\tilde\phi}$ have unip constituents lying in the same family.



$* — \bullet \iff \langle *, \bullet \rangle \neq 0$

in this sense, joint / equiv. iff connected

$\{1,2\}$, $\{3,4,5\}$ are families

$\updownarrow$ a $\qquad$ $\updownarrow$ $b \cup c$

Prop 21. let σ be ordinary

equiv. classes in $\mathrm{Irr}(W)^\sigma$ are precisely σ-stable families in the sense of Def 17

Prop 22.

let $\left| \begin{array}{l} \mathcal{U} \subset \mathrm{Ud}(G^F) \text{ be a family} \\ \mathcal{F} \subset \mathrm{Irr}(W)^\sigma \text{ be a "corresponding" family} \end{array} \right.$

then a (resp. A) funct is constant on $\mathcal{U}$ and $\mathcal{F}$ and their values agree

$\uparrow$ Def 7 $\qquad$ $\uparrow$ Def 12+14

Prop 23.

D_G sends families in $\mathrm{Ud}(G^F)$ to families.

V. Fourier transform

Def 24. let H be a finite gp.

$$\mathcal{M}(H) := \{(x,\sigma) \in H \times \mathrm{Irr}(C_H(x))\} / \text{conj}.$$

$$\{\,,\,\} : \mathcal{M}(H) \times \mathcal{M}(H) \longrightarrow \mathbb{C}$$

$$((x,\sigma),(y,\tau)) \longmapsto \frac{1}{|C_H(x)|\,|C_H(y)|} \sum_{\substack{g\in H \\ gyg^{-1}\in C_H(x)}} \sigma(gyg^{-1})\,\tau(g^{-1}x^{-1}g)$$

Def: Non-abelian Fourier transform associated to H:

$$M_H : \mathcal{F}(\mathcal{M}(H),\mathbb{C}) \longrightarrow \mathcal{F}(\mathcal{M}(H),\mathbb{C})$$

$$f \longmapsto \left((x,\sigma) \mapsto \sum_{(y,\tau)\in\mathcal{M}(H)} \{(x,\sigma),(y,\tau)\}\, f(y,\tau) \right)$$

Prop 25.

- $\{(x,\sigma),(y,\tau)\} = \overline{\{(y,\tau),(x,\sigma)\}}$

(* $M_H^2 = \mathrm{id}$)

- $\mathcal{M}(H) \xleftrightarrow{1-1}$ iso. classes of simple H-graded $\mathbb{C}H$-mod.

Twisted version :

$H \lhd \tilde{H}$ where $[\tilde{H} : H] = c$, it is a split cyclic extension.

fix $\tilde{H}/H = \langle x_0 \rangle$ and $H' := x_0 H$

Def 26.

$$\mathcal{M}(H \lhd \tilde{H}) := \left\{ (x,\sigma) \in H \times \mathrm{Irr}(C_{\tilde{H}}(x)) \;\middle|\; \begin{array}{l} H' \cap C_{\tilde{H}}(x) \neq \emptyset \\ (x,\sigma) \in \mathcal{M}(H) \end{array} \right\} / \tilde{H}\text{-conj.}$$

$$\bar{\mathcal{M}}(H \lhd \tilde{H}) := \left\{ (x,\bar{\sigma}) \in H' \times \mathrm{Irr}(C_H(x)) \right\} / \tilde{H}\text{-conj.}$$

Rmk

* $\mathrm{Irr}(\tilde{H}/H)$ is cyclic and acts on $\mathcal{M}(H \lhd \tilde{H})$ via:

  $\chi \cdot (x,\sigma) := (x, \sigma \otimes \chi)$ ⇝ denote R a sys of rep.

* $C_{\tilde{H}}(x)$ is a central cyclic ext. of $C_H(x)$ by x_0

  $\Rightarrow$ irred char. of $C_H(x)$ extend to $C_{\tilde{H}}(x)$

  denote $\bar{\sigma} \rightsquigarrow \sigma$.

Def 27.

$$\{\,,\,\} : \bar{\mathcal{M}}(H \lhd \tilde{H}) \times \mathcal{M}(H \lhd \tilde{H}) \longrightarrow \mathbb{C}$$

$$\{(x,\bar{\sigma}),(y,\tau)\} \longmapsto c\,\{(x,\sigma),(y,\tau)\}$$

Def 28.

$$M_{H\le\tilde H} : \mathcal{F}(\mathcal{M}(H\triangleleft\tilde H), \mathbb{C}) \to \mathcal{F}(\bar{\mathcal{M}}(H\triangleleft\tilde H), \mathbb{C})$$

$$f \mapsto \Big((x,\bar\sigma) \mapsto \sum_{(y,\tau)\in R} \{(x,\bar\sigma),(y,\tau)\}\, f(y,\tau) \Big)$$

Prop 29.

$M_{H\le\tilde H}$ is an iso.

Applicat°: Assume that σ is ordinary.

let $\mathcal{U} \subset \mathrm{Uch}(G^F) \longleftrightarrow \mathcal{F} \subset \mathrm{Irr}(W)^{\sigma}$

Lusztig: constructed $G_{\mathcal{U}} \triangleleft \tilde G_{\mathcal{U}}$ s.t. $[\tilde G_{\mathcal{U}} : G_{\mathcal{U}}] = \mathrm{ord}(\sigma)$.

and $\mathcal{U} \xhookrightarrow{1-1} \bar{\mathcal{M}}(G_{\mathcal{U}} \triangleleft \tilde G_{\mathcal{U}})$

$\rho \longmapsto \bar x_\rho$

and $\mathcal{F} \hookrightarrow \mathcal{M}(G_{\mathcal{U}} \triangleleft \tilde G_{\mathcal{U}})$

$\phi \longmapsto x_\phi$

$G_{\mathcal{U}}$ is a nice quotient of $\pi_0(C_G(u)/Z_G)$ where u is a special element.

Def 30

let $\rho\in\mathrm{Uch}(G^F)$ occuring in $\mathrm{Ind}_L^G\,\theta$ with θ cuspidal unip.

def. $\Delta(\rho) := \varepsilon_L = (-1)^{\mathrm{rk}_{\mathbb{F}_q}(L)}$

Thm 31

$\forall \rho\in\mathcal{U}$, $\phi\in\mathrm{Irr}(W)^{\sigma}$ with preferred σ-extens° $\tilde\phi$,

$$\langle \rho, R_{\tilde\phi}\rangle = \Delta(\rho)\{\bar x_\rho, x_\phi\}$$

Cor 32

$$\forall \Phi \in \mathcal{F}, \quad R_{\tilde{\Phi}} = \sum_{\rho \in \mathcal{U}} \Delta(\rho) \langle \bar{\alpha}_\rho, \alpha_\Phi \} \rho .$$

Thm 33

Let G be a complete root datum.

There exists | a set $\mathrm{Uch}(G)$ s.t.
| a funct° $\mathbb{D} : \mathrm{Uch}(G) \longrightarrow \mathbb{Q}[q]$

$$\gamma \longmapsto \mathbb{D}_\gamma$$

for any admissible prime power q (i.e. a pair (G, F)),

$\exists$ bij. $\psi_q^G : \mathrm{Uch}(G) \longrightarrow \mathrm{Uch}(G^F) : \mathbb{D}_\gamma = \mathbb{D}_{\psi_q^G(\gamma)}$

VI. Examples

1. Type A.

partition of n

$$\begin{tikzcd} W = S_n , & \mathrm{Irr}(W) \arrow[r, hook, "1-1"] \arrow[d, leftrightarrow, "1-1"'] & \mathrm{Conj}(W) \arrow[r, hook, "1-1"] & \{\alpha = (\alpha_1 \leq \cdots \leq \alpha_m) \vdash n\} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ & \mathrm{Uch}(G) \ni \rho_\alpha & & \alpha \arrow[ll, mapsto] \end{tikzcd}$$

Prop 34. Hook formula for $\sigma = \mathrm{id}$ (PGL_n)

$$\mathbb{D}_{\rho_\alpha} = q^{a(\alpha)} \prod_{i=1}^{n} \frac{q^i - 1}{q^{l_i} - 1}$$

with | $a(\alpha) := \sum_{i=1}^{m} (m-i)\alpha_i$
| $l_i :=$ length of the hook at the i-th box of Young diag of α

Prop 35 $\sigma = id$. (PGL_n).

$$\mathbb{D}_{\rho_\alpha} = \frac{\prod_{i=1}^{n}(q^i-1)\prod_{i<j}(q^{\lambda_j}-q^{\lambda_i})}{q^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots+1}\prod_{i=1}^{n}\prod_{k=1}^{\lambda_i}(q^k-1)} \qquad \text{with } \lambda_i = \alpha_i + i - 1$$

Example:

$*$ $n=2$

α	$\mathbb{D}_{\rho_\alpha}$
(2)	1
$(1\,1)$	q

$*$ $n=3$

α	$\mathbb{D}_{\rho_\alpha}$
(3)	1
$(2\,1)$	$q\Phi_2$
$(1\,1\,1)$	q^3

cyclotomic polynomial

$*$ $a_\rho = 0 \iff \rho = 1$

$A_\rho = \ell(w_0) \iff \rho = St$

Prop 36. Ennola duality.

if $\sigma \neq id$. (PSU_n)

then $\forall \gamma_\alpha \in Ud(PSU_n^F)$, $\mathbb{D}_{\gamma_\alpha}(q) = \mathbb{D}_{\rho_\alpha}(-q)$

2. Type B, C, D

$$W_{B_n} \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \underset{\text{index } 2}{\supset} W_{D_n}$$

Def 37. Lusztig symbols.

let $a, b \in \mathbb{N}$, $d \in \mathbb{Z}$.

def the set of symbols:

$$\tilde{\mathcal{X}}^{a,b}_d := \left\{ S = \binom{X}{Y} \;\middle|\; \begin{matrix} X = (x_1, \dots, x_r) \\ Y = (y_1, \dots, y_s) \end{matrix}, \quad \begin{matrix} r - s = d \\ y_1 \geq b \\ x_j \geq x_{j-1} + a + b \\ y_j \geq y_{j-1} + a + b \end{matrix} \right\}$$

set / tuples

* d is called the defect

* rank: $\mathrm{rk}(S) := \sum_{x \in S} x - (a+b) \left\lfloor \frac{(r+s-1)^2}{4} \right\rfloor - b \left\lfloor \frac{r+s}{2} \right\rfloor$

* $S' := \begin{pmatrix} 0 & x_1 + a + b & \cdots & x_r + a + b \\ b & y_1 + a + b & \cdots & y_s + a + b \end{pmatrix} \in \tilde{\mathcal{X}}^{a,b}_d$ shift operation

* $\mathcal{X}^{a,b}_d := \tilde{\mathcal{X}}^{a,b}_d /$ shift operat°

* $\mathcal{X}^{a,b}_{d,n} := \{$ rank n symbols in $\mathcal{X}^{a,b}_d \}$

Example:

$$\mathcal{X}^{0,0}_{d,n} \xhookrightarrow{1-1} \{ \text{bipartition } (\alpha, \beta) \vdash n \} \xhookrightarrow{1-1} \mathrm{Irr}(W_{B_n}).$$

Def 38.

$S = \binom{X}{Y} \longmapsto S^{tr} = \binom{Y}{X}$ def an act° on $\mathcal{X}_d^{a,0} \cup \mathcal{X}_{-d}^{a,0}$.

we denote $\mathcal{Y}_d^a$ the set of equiv. classes under this act° and $\mathcal{Y}_{d,n}^a$ the subset of elem^T of rank n.

Thm 39.

- $d \equiv 1\,[2] \overset{\text{type } B_n}{\Longrightarrow} \mathcal{U}d(G) \overset{1-1}{\hookrightarrow} \mathcal{Y}^1_{d,n}$.
- G of type D_n, $d \equiv 0\,[4] \overset{d \neq 0}{\Longrightarrow} \mathcal{U}d(G) \overset{1-1}{\hookrightarrow} \mathcal{Y}^1_{d,n}$
- ———— ${}^2D_n$, $d \equiv 2\,[4]$ ————

Prop 40 Hook formula SU_{2n+1}, Sp_{2n}, SO_{2n}.

$[S] \in \mathcal{Y}^1_{d,n}$ for admissible d.

$$\mathbb{D}_{\rho_S} = q^{a(S)} \frac{|G|_{q'}}{2^{b'(S)} \prod_{\text{hook } h}(q^{\ell(h)}-1) \prod_{\substack{\text{cohook} \\ c}}(q^{\ell(c)}-1)}$$

with

$$a(S) := \sum_{\{x,y\} \subset S} \min\{x,y\} - \sum_{i \geq 1} \binom{\pi+s-2i}{2}$$

$$b'(S) := \begin{cases} \left\lfloor \frac{\pi+s-1}{2} \right\rfloor - |X \cap Y| & \text{if } X \neq Y \\ 0 & \text{else} \end{cases}$$

Prop 4.1

$$S=\binom{X}{Y}=\begin{pmatrix}x_1<\cdots<x_r\\ y_1<\cdots<y_s\end{pmatrix}$$

$$\mathbb{D}_{\rho_S}\doteq\frac{|G|_{q'}\prod_{i<j}(q^{x_j}-q^{x_i})(q^{y_j}-q^{y_i})\prod_{i,j}(q^{x_i}+q^{y_j})}{2^{b(S)}\,q^{\binom{r+s-2}{2}+\binom{r+s-4}{2}+\cdots+1}\prod_{i=1}^{r}\prod_{k=1}^{x_i}(q^{2k}-1)\prod_{j=1}^{s}\prod_{l=1}^{y_j}(q^{2l}-1)}$$

with $b(S):=\begin{cases}\lfloor\frac{r+s-1}{2}\rfloor & \text{if } X\neq Y\\ r & \text{else}\end{cases}$

Prop 4.2. ρ_{S_1} and ρ_{S_2} in the same family $\Longleftrightarrow$ $X_1\cup Y_1=X_2\cup Y_2$ counting with multiplicity

Example: $\mathcal{G}'_{1,2}$, Sp_4

S	$\mathbb{D}_{\rho_S}$	$\mathrm{Irr}(W)$	
$\binom{2}{\ }$	1	$(2,-)$	$\mathbb{1}$
$\begin{pmatrix}0&2\\ &1\end{pmatrix}$	$\frac{1}{2}q\Phi_2^2$	$(1,1)$	same family $a=1$ $A=3$
$\begin{pmatrix}1&2\\ &0\end{pmatrix}$	$\frac{1}{2}q\Phi_4$	$(11,-)$	
$\begin{pmatrix}0&1\\ &2\end{pmatrix}$	$\frac{1}{2}q\Phi_4$	$(-,2)$	
$(0\ 1\ 2)$	$\frac{1}{2}q\Phi_1^2$	cuspidal $2=1^2+1$	
$\begin{pmatrix}0&1&2\\ &1&2\end{pmatrix}$	q^4	$(-,11)$	St.

$$\begin{pmatrix}\alpha_0 & \alpha_1+1 & \underline{\alpha_2+2}\\ \beta_0 & \underline{\beta_1+1} & \end{pmatrix}\qquad(\alpha_0\ \alpha_1\ \underline{\alpha_2},\ \beta_0\ \underline{\beta_1})$$