

With vectors

We start the talk with some notations, clarifications and recollections.

Notation 1

- (1) p is a fixed prime number
- (2) $E/\mathbb{Q}_p$ is a finite field extension. Since this extension is finite, the p -adic norm on $\mathbb{Q}_p$ uniquely extends to a norm on E .
- (3) $\mathcal{O}_E = \{e \in E \mid |e| \leq 1\} \subseteq E$ ring of integers.
- (4) $\mathcal{O}_E$ is a local ring with $\{e \in E \mid |e| < 1\} = \mathfrak{m}_E$ maximal. This maximal ideal is principal, so we fix some element $\pi \in \mathfrak{m}_E$ with $(\pi) = \{e \in E \mid |e| < 1\}$. We call π a uniformizer.

(5) $\mathcal{O}_E/(\pi)$ is called the residue field of $\mathcal{O}_E$. It is a finite field extension of $\mathbb{F}_p$, so there $\exists q \in \mathbb{P}^{\mathbb{N}}$ with $\mathcal{O}_E/(\pi) = \mathbb{F}_q$.

(6) If A is an $\mathcal{O}_E$ -Algebra, we can view π as an element of A via the structure morphism $\mathcal{O}_E \longrightarrow A$.

(7) We call an $\mathcal{O}_E$ -algebra π -complete, if

$$A \longrightarrow \varprojlim_{n \in \mathbb{N}^{\geq 1}} A/\pi^n$$

is an isomorphism of $\mathcal{O}_E$ -algebras.

(8) We call an $\mathcal{O}_E$ -algebra A π -torsion free, if

$A \xrightarrow{\cdot \pi} A$ is injective.

The goal of this talk is to introduce the ring of Witt vectors $W_{O_E, \pi}(A)$ for a O_E -algebra A .

To make calculations nicer, there is the following lemma:

Lemma 2

Let A be a O_E -algebra, $I \subseteq A$ an ideal such that $\pi \in I$. We have the statement:

$$a \equiv b \pmod{I} \implies a^{q^k} \equiv b^{q^k} \pmod{I^{k+1}} \quad \forall k \geq 0$$

Proof:

See [Auschnitz, Lemma 3.1].

Definition 3 (Witt polynomials)

For $n \in \mathbb{N}$ we define the n -th Witt polynomial

$$W_{n, \pi} := \sum_{i=0}^n \pi^i X_i^{q^{n-i}} \in O_E[X_0, \dots, X_n]$$

For $E := \mathbb{Q}_p, \pi := p$, those are the "classical" Witt polynomials.

Lemma 4

There $\exists$ a unique functor

$$W_{O_E, \pi} : (O_E\text{-Alg}) \longrightarrow (O_E\text{-Alg})$$

such that

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A^{\mathbb{N}} \\ (O_E\text{-Alg}) & \xrightarrow{\quad} & (\text{Set}) \\ W_{O_E, \pi} \downarrow \text{dashed} & \nearrow \text{forget} & \\ (O_E\text{-Alg}) & & \end{array}$$

commutes and such that $\forall O_E\text{-algebra } A$ the natural map

$$\begin{aligned} W_{\pi, A} : W_{O_E, \pi}(A) &\longrightarrow A^{\mathbb{N}} \\ (a_0, a_1, \dots) &\longmapsto (W_{n, \pi}(a_0, \dots, a_n))_{n \geq 0} \end{aligned}$$

is a morphism of O_E -algebras.

Note: By commutativity of the diagram it's obvious that $W_{O_E, \pi}(A) = A^{\mathbb{N}}$ as sets. So proving the

statement means that we need to find a suitable ring structure on $W_{\mathcal{O}_E, \pi}(A) = A^{\mathbb{N}}$.

Proof:

Step 1:

Let A be an π -torsion free $\mathcal{O}_E$ -algebra with a $\mathcal{O}_E$ -Morphism $\ell: A \rightarrow A$ such that $\ell(a) \equiv a^q \pmod{\pi}$.
(Frobenius lift)

We claim that the map

$$W_{\pi, A}: W_{\mathcal{O}_E, \pi}(A) \longrightarrow A^{\mathbb{N}}$$

$$(a_n)_{n \geq 0} \longmapsto (W_{\pi, n}(a_1, \dots, a_n))_{n \geq 0}$$

is injective with image

$$M := \left\{ (b_n)_{n \geq 0} \in A^{\mathbb{N}} \mid b_{n+1} \equiv \ell(b_n) \pmod{\pi^{n+1}} \right\}$$

As ℓ is a $\mathcal{O}_E$ -morphism, we see that M is closed under componentwise addition and multiplication. We equip $W_{\mathcal{O}_E, \pi}(A)$ with the

$\mathcal{O}_E$ -algebra structure from M via the injection.
 $W_{\Pi, A}$ then automatically becomes an $\mathcal{O}_E$ -map.
The uniqueness part is also clear by this definition.

We prove the claim:

1) Injectivity:

Suppose $W_{\Pi, A}((a_n)_n) = W_{\Pi, A}((b_n)_n)$.

We prove $(a_n)_n = (b_n)_n$ by induction.

- For $n=0$:

$$\begin{aligned} a_0 &= W_{0, \Pi}(a_0) = W_{\Pi, A}((a_n)_n)_0 = W_{\Pi, A}((b_n)_n)_0 \\ &= W_{0, \Pi}(b_0) = b_0 \end{aligned}$$

- Suppose we know $(a_n)_{n \leq m} = (b_n)_{n \leq m}$

We have

$$W_{m+1, \pi}(a_1, \dots, a_{m+1}) = W_{m+1, \pi}(b_1, \dots, b_{m+1}).$$

$$\parallel \sum_{i=0}^{m+1} \pi^i \cdot a_i q^{m+1-i}$$

$$\parallel \sum_{i=0}^{m+1} \pi^i \cdot b_i q^{m+1-i}$$

$$\Rightarrow \pi^{m+1} \cdot a_{m+1} = \pi^{m+1} \cdot b_{m+1}$$

$$\Rightarrow a_{m+1} = b_{m+1}$$

$\uparrow$
A is π -torsion free

$$(2) \underline{\text{Im}(W_{\pi, A})} \leq M.$$

We calculate mod π^{n+1} :

$$\varphi(W_{n, \pi}(a)) \underset{\substack{\uparrow \\ \varphi \text{ is OE-morph.}}}{=} \sum_{i=0}^n \pi^i \cdot \varphi(a_i) q^{n-i}$$

$$\begin{aligned} & \left(\begin{array}{l} \varphi(a_i) \equiv a_i q^{n-i} \pmod{\pi} \\ \Downarrow \text{Lemma 2} \\ \varphi(a_i) q^{n-i} \equiv a_i q^{n-i} \pmod{\pi} \\ \Downarrow \\ \pi^i \varphi(a_i) q^{n-i} \equiv \pi^i a_i q^{n-i} \pmod{\pi^{n+1}} \end{array} \right) \xrightarrow{\text{Lemma 2}} \sum_{i=0}^n \pi^i \cdot a_i q^{n+1-i} \\ & \equiv \sum_{i=0}^n \pi^i \cdot a_i q^{n+1-i} + \pi^{n+1} \cdot a_{n+1} \\ & = \sum_{i=0}^{n+1} \pi^i \cdot a_i q^{n+1-i} = W_{n+1, \pi}(a) \pmod{\pi^{n+1}} \end{aligned}$$

(3) $M \subseteq \mathcal{I}_m(W_{\pi, A})$:

Let $(b_n)_{n \geq 0} \in M$. We have to find $(a_n)_{n \geq 0} \in A^{\mathbb{N}}$ such that $(b_n)_{n \geq 0} = (W_{n, \pi}(a_0, \dots, a_n))_{n \geq 0}$.

We define $a_0 := b_0$. We assume that $a_0, \dots, a_m$ are already defined such that $(b_n)_{n \leq m} = (W_{n, \pi}(a_0, \dots, a_n))_{n \leq m}$.

We need to solve the equation

$$b_{m+1} = \sum_{i=0}^{m+1} \pi^i \cdot a_i^{q^{m+1-i}},$$

for a_{m+1} . Since we have

$$\underbrace{b_{m+1} - \sum_{i=0}^m \pi^i \cdot a_i^{m+1-i}}_{(*)} = \pi^{m+1} \cdot a_{m+1}$$

we need to show that $(*) \equiv 0 \pmod{\pi^{m+1}}$.

We calculate mod π^{m+1} :

$$b_{m+1} \stackrel{b \in M}{=} f(b_m) \stackrel{\text{Def. ran } a_0, \dots, a_m}{=} f(W_{m,\pi}(a_0, \dots, a_m))$$

$$\stackrel{f \text{ is } \mathcal{O}_E\text{-morph.}}{=} \sum_{i=0}^m \pi^i \cdot f(a_i)^{q^{m-i}}$$

$$\stackrel{\text{Lemma 2}}{=} \sum_{i=0}^m \pi^i \cdot a_i^{q^{m+1-i}}$$

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$$\text{so } (\#) \equiv 0 \pmod{\pi^{m+1}}$$

Step 2:

Let $A := \mathcal{O}_E[X_1, \dots, X_n]$ be a polynomial ring.

Then A is obviously π -torsion free. Also there is a

Frobenius lift:

$$\begin{array}{ccc} \varphi: \mathcal{O}_E[X_1, \dots, X_n] & \longrightarrow & \mathcal{O}_E[X_1, \dots, X_n] \\ X_i & \longmapsto & X_i^q \end{array}$$

and we have $\varphi(a) = a^q \bmod \pi$, since $(\cdot)^q$ is additive on $\mathcal{O}_E/\pi[X_1, \dots, X_n] = \mathbb{F}_q[X_1, \dots, X_n]$.
So by Step 1, the statement is true for A .

Step 3:

Extend the definition of $W(\cdot)$ to quotients $\mathcal{O}_E[X_1, \dots, X_n]/I$.

Step 4:

Let A be a $\mathcal{O}_E$ -algebra. Write $A = \operatorname{colim}_\lambda A_\lambda$ for some $\mathcal{O}_E$ -algebras A_λ of finite presentation [Stacks Project, 002N].
Use Step 3 and extend the definition of $W(\cdot)$ to colimits. $\square$

Uniqueness of $W_{O_E, \pi}$:

Let A be a O_E -algebra, π -torsion free with a Frobenius lift. It can be seen in Step 1 that $W_{\pi, A}$ is injective and that $\text{Im}(W_{\pi, A})$ is a O_E -subalgebra of A^N . This makes the O_E -algebra structure on $W_{O_E, \pi}(A)$ unique such that $W_{\pi, A}$ is a morphism of O_E -algebras.

For arbitrary B the object $O_{E, \pi}(B)$ is automatically defined, because we can write B as a colimit of quotients of O_E -algebras A that are π -torsion free with a Frobenius lift.

□

Definition 5

Let A be a $\mathcal{O}_E$ -algebra. By lemma 4 there is!
a $\mathcal{O}_E$ -algebra $W_{\mathcal{O}_E, \pi}(A)$ such that $W_{\mathcal{O}_E, \pi}(A) = A^{\mathbb{N}}$
as sets and such that

$$W_{\pi, A}: W_{\mathcal{O}_E, \pi}(A) \longrightarrow A^{\mathbb{N}}$$
$$(a_n)_{n \geq 0} \longmapsto (W_{n, \pi}(a_0, \dots, a_n))_{n \geq 0}$$

is a morphism of $\mathcal{O}_E$ -algebras. We call $W_{\mathcal{O}_E, \pi}(A)$
the ring of ramified Witt vectors of A .

Remark 6

Warning: The statement of Lemma 4 is not that the $\mathcal{O}_E$ -algebra structure on $W_{\mathcal{O}_E, \pi}(A) = A^{\mathbb{N}}$ is unique such that $W_{\pi, A}$ is a morphism of $\mathcal{O}_E$ -algebras. $\mathbb{F}_p = \mathbb{Z}_p/p$ is a $\mathbb{Z}_p$ -algebra

Example: $\mathcal{O}_E = \mathbb{Q}_p$, $\pi = p$, $A = \overline{\mathbb{F}_p}$. It is known that $W_{\mathcal{O}_E, \pi}(A) = \mathbb{Z}_p$ (we will see later). Consider the map

$$W: \overline{\mathbb{F}_p}^{\mathbb{N}} \longrightarrow \overline{\mathbb{F}_p}^{\mathbb{N}} \\ (a_n)_{n \geq 0} \longmapsto (W_{p, n}(a_n)_{n \geq 0})$$

In this case the map is really simple, because $p = 0$ and $a^p = a$ in $\overline{\mathbb{F}_p}$.

$$W((a_n)_{n \geq 0}) = (a_0, a_0^p, a_0^{p^2}, \dots) = (a_0, a_0, a_0, \dots)$$

This is a morphism of $\mathbb{Z}_p$ -algebras, as $\mathbb{F}_p^{1/N}$ and $\mathbb{Z}_p$ both have the desired property.

So why $W_{E,\pi}(A) = \mathbb{Z}_p$ and not $W_{E,\pi}(A) = \mathbb{F}_p^{1/N}$?

The statement of Lemma 4 is that the functor

$W_{E,\pi}$ is unique with certain properties. It can be

seen in the proof of Lemma 4 that the

$\mathbb{Q}_E$ -algebra structure of $W_{E,\pi}(A)$ is unique such that

$W_{\pi,A}$ is a morphism at least for $\mathbb{Q}_E$ -algebras

A that are π -torsion free and have a Frobenius lift.

Example 7

We calculate how "+" works in $W_{0E, \Pi}(A)$:

Suppose

$$(a_n)_n + (b_n)_n = (c_n)_n$$

in $W_{0E, \Pi}(A)$. We want to calculate c_n for small n . We use that $W_{\Pi, A}$ is additive.

$$W_{\Pi, A}((a_n)_n + (b_n)_n) = W_{\Pi, A}((a_n)_n) + W_{\Pi, A}((b_n)_n)$$

$\parallel \qquad \qquad \qquad \parallel$

$$W_{\Pi, A}((c_n)_n) \qquad (W_{n, \Pi}(a_0, \dots, a_n))_n + (W_{n, \Pi}(b_0, \dots, b_n))_n$$

$\parallel$

$$(W_{n, \Pi}(c_0, \dots, c_n))_n$$

$$\underline{h=0:}$$

$$C_0 = a_0 + b_0$$

$$\underline{h=1:}$$

$$C_0^q + \pi \cdot C_1 = (a_0^q + \pi \cdot a_1) + (b_0^q + \pi \cdot b_1)$$

$$C_0 = a_0 + b_0$$

$$\Rightarrow C_1 = \frac{a_0^q + b_0^q - (a_0 + b_0)^q}{\pi} + a_1 + b_1$$

Proposition 8

Let A be a $\mathcal{O}_E$ -algebra.

(1) The map

$$\begin{aligned} [\cdot] : A &\longrightarrow W_{0,E,\pi}(A) \\ a &\longmapsto [a] = (a, 0, 0, \dots) \end{aligned}$$

is multiplicative. We call it Teichmüller lift.

(2) $\exists$ ring homomorphism

$$F : W_{0,E,\pi}(A) \longrightarrow W_{0,E,\pi}(A)$$

lifting the q -Frobenius ($F(a) = a^q \bmod \pi$).

If $\pi \cdot A = 0$, then $\exists$ a q -Frobenius $\ell : A \xrightarrow{(\cdot)^q} A$ on $A = A/\pi$ and we have $F = W_{0,E,\pi}(\ell)$.

(3) The map

$$\begin{aligned} V_\pi : W_{0,E,\pi}(A) &\longrightarrow W_{0,E,\pi}(A) \\ (a_0, a_1, \dots) &\longmapsto (0, a_0, a_1, \dots) \end{aligned}$$

is $\mathbb{Q}_E$ -linear and satisfies

- $F \circ V_\pi = \pi \cdot (\cdot)$
- $V_\pi (F(x) \cdot y) = x \cdot V_\pi (y)$

(4) We have an isomorphism

$$W_{\mathbb{Q}_E, \pi}(A) \xrightarrow{\sim} \varprojlim_n W_{\mathbb{Q}_E, \pi}(A) / \underbrace{V_\pi \circ \dots \circ V_\pi}_{n\text{-times}} (W_{\mathbb{Q}_E, \pi}(A))$$

and every $a \in W_{\mathbb{Q}_E, \pi}(A)$ can be written uniquely in the form

$$a = \sum_{n \geq 0} V_\pi^n([a_n])$$

for some $a_n \in A$, $n \geq 0$.

(5) If $\pi \cdot A = 0$, then $V_\pi \circ F = \pi \cdot (\cdot)$
and $F([a]) = [a^q]$.

(6) If $\pi A = 0$ and A is perfect

(the Frobenius $A \rightarrow A, a \mapsto a^p$ is bijective. Note that $(\cdot)^p$ is a ring morphism, since $A = A/\pi$), then

- $\bigvee_{\pi}^n W_{0E, \pi}(A) = \pi^n \cdot W_{0E, \pi}(A)$
- $W_{0E, \pi}(A)$ is π -complete
- $W_{0E, \pi}(A)$ is π -torsion free
- $W_{0E, \pi}(A)/\pi \cong A$
- Every $a \in W_{0E, \pi}(A)$ can be written uniquely as

$$a = \sum_{n \geq 0} \pi^n \cdot [a_n]$$

for some $a_n \in A, n \geq 0$.

Example 9

$E := \mathbb{Q}_p$, $A := \mathbb{F}_p$, $\pi := p$, then we can use Proposition 8 (6) to find an isomorphism

$$W_{\mathbb{Z}_p, p}(\mathbb{F}_p) \xrightarrow{\sim} \mathbb{Z}_p$$
$$\sum_{n \geq 0} p^n [a_n] \qquad \sum_{n \geq 0} p^n b_n$$

where $b_n \in \{0, \dots, p-1\}$, $b_n \equiv a_n \pmod{p}$. It is not clear that this is an isomorphism, so this needs to be checked.

Lemma 10

$$\begin{array}{c} \mathcal{O}_E/\pi \\ \parallel \end{array}$$

Let A be a perfect $\mathbb{F}_q$ -algebra and let $E_0 \subseteq E$ be the maximal unramified subextension of E .
Then $\exists$ an isomorphism

$$W_{\mathbb{Z}_p, p}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \cong W_{\mathcal{O}_{E_0}, \pi}(A)$$

of $\mathcal{O}_E$ -algebras.

Remark

- $\mathcal{O}_{\mathbb{F}_p} = \mathbb{Z}_p$, so $W_{\mathbb{Z}_p, p}$ makes sense.
- A is a $\mathcal{O}_E$ -algebra via $\mathcal{O}_E \rightarrow \mathcal{O}_E/\pi = \mathbb{F}_q \rightarrow A$,
so $W_{\mathcal{O}_{E_0}, \pi}(A)$ makes sense.
- A is an $\underbrace{\mathcal{O}_{\mathbb{F}_p}}_{\mathbb{Z}_p}$ -algebra via $\mathcal{O}_{\mathbb{F}_p} \rightarrow \mathcal{O}_E \rightarrow A$,
so $W_{\mathbb{Z}_p, p}(A)$ makes sense.
- Definition of 'unramified':

Let $K/\mathbb{Q}$ be a finite extension. For $\alpha \in K$ define $\text{ord}(\alpha) := -\log_p(|\alpha|)$. One can show $\text{Im}(\text{ord}(\cdot)) = \frac{1}{e} \cdot \mathbb{Z}$ for some $e \in \mathbb{N}$.

We say $K/\mathbb{Q}$ is unramified, if $e = 1$.

There $\exists$ unramified extension $K_f^{\text{unram}}/\mathbb{Q}_p$ of degree f . See [Koblitz, *p*-adic Analysis, III.3]

- One can show $W_{\mathbb{Z}_p, p}(\mathbb{F}_q) = \mathcal{O}_{E_0}$. So

$W_{\mathbb{Z}_p, p}(A)$ is a $\mathcal{O}_{E_0}$ -algebra via

$\mathcal{O}_E = W_{\mathbb{Z}_p, p}(\mathbb{F}_q) \longrightarrow W_{\mathbb{Z}_p, p}(A)$. So the $\otimes$ makes sense.

Proof ideas

(1) By Proposition 8 (6) we have that $W_{\mathcal{O}_E, \pi}(A)$ is

π -adically complete and $W_{\mathcal{O}_E, \pi}(A)/\pi \cong A$.

(2) By Proposition 8 (6) we have that $W_{\mathbb{Z}_p, p}(A)$ is p -adically complete and $W_{\mathbb{Z}_p, p}(A)/p \cong A$.

Now show that $(W_{\mathbb{Z}_p, p}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)/\pi \cong A$. (2)

(3) Define morphism

$$W_{\mathbb{Z}_p, p}(A) \longrightarrow W_{\mathcal{O}_E, \pi}(A)$$

$$\sum_{n \geq 0} p^n \cdot [a_n] \longmapsto \sum_{n \geq 0} \pi^{en} \cdot \varepsilon^n \cdot [a_n]$$

where ε and e are chosen st. $p = \varepsilon \cdot \pi^e$. Well defined?

This induces a morphism of $\mathcal{O}_E$ -algebras (?)

$$W_{\mathbb{Z}_p, p}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \longrightarrow W_{\mathcal{O}_E, \pi}(A)$$

We have to show that this is an isomorphism (?)

(4) Idea why they are isomorphic:

Both rings are π -adically complete, so

$$W_{\mathbb{Z}_{p,p}}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \longrightarrow W_{\mathcal{O}_{E,\pi}}(A)$$

$$\parallel \qquad \parallel$$

$$\varprojlim_{n \geq 1} \frac{W_{\mathbb{Z}_{p,p}}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E}{\pi^n} \qquad \varprojlim_n \frac{W_{\mathcal{O}_{E,\pi}}(A)}{\pi^n}$$

and at least for $n=1$ we know

$$\frac{W_{\mathbb{Z}_{p,p}}(A) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E}{\pi^1} = A = \frac{\varprojlim_n W_{\mathcal{O}_{E,\pi}}(A)}{\pi^1}$$

What about $n \geq 2$? Is this compatible with our potential isomorphism?

Definition 11

(1) A perfect prism over $\mathcal{O}_E$ is a pair

$$(W_{\mathcal{O}_E, \pi}(R), I)$$

with

- R a perfect $\mathbb{F}_q$ -algebra
- $I \subseteq W_{\mathcal{O}_E, \pi}(R)$ an ideal generated by some $d \in W_{\mathcal{O}_E, \pi}(R) \times \mathbb{A}^\times$.

$$\frac{F(d) - d^q}{\pi} \in W_{\mathcal{O}_E, \pi}(R)^\#$$

Here F is defined in Prop. 8 (2).

- $W_{\mathcal{O}_E, \pi}(R)$ is I -adically complete.

(2) An $\mathcal{O}_E$ -algebra A is perfectoid if

$$A \cong W_{\mathcal{O}_E, \pi}(R)/I$$

for some perfect prism $(W_{\mathcal{O}_E, \pi}(R), I)$ over $\mathcal{O}_E$.

Proposition 12

Let S be a ring and $w \in S$ a non-zero divisor such that

- $w^p \mid p$
- S is w -adically complete
- $S/w \cdot S \xrightarrow{(\)^p} S/w^p \cdot S$ is an iso.

Then S is perfectoid.