Grothendieck Katz Conjecture

Let

$$\frac{df}{dz} = A(z)f$$

be a linear ordinary differential equation, where $A(z)$ is an $n \times n$-matrix with entries in $\mathbb{Q}(z)$. Then the solutions form an n-dimensional $\mathbb{C}$-vector space, if the differential equation defines an integrable connection

$$\nabla(f) = df - A(z)dz * f.$$

The solutions are holomorphic functions and we ask ourselves: When are all solutions of the differential equation even algebraic functions?

The Grothendieck Katz conjecture proposes: It has a full set of algebraic solutions if and only if the reduced equation modulo p has a full set of solutions for almost all p.

The Grothendieck Katz conjecture is formulated in a greater generality. We want to make sense of a flat connection having a full set of algebraic solutions. Let S be a smooth connected $\mathbb{C}$-scheme and (M, ∇), where M is a locally free sheaf of finite rank r and ∇ an integrable connection on M. Then we say that (M, ∇) *has a full set of algebraic solutions*

- $\Leftrightarrow$ $\exists$ finite field extention K/L, where L is the function field of S, such that $(M, \nabla) \otimes K$ is trivial (i.e. $M \otimes K$ is spanned by horizontal sections, $M \otimes K \cong (M \otimes K)^{\nabla} \otimes K$
- $\Leftrightarrow$ $\exists$ a finite cover $u' : U' \to S$ such that $u'^{*}(M, \nabla) \cong (\mathcal{O}_{U'}^r, d)$ the trivial connection
- $\Leftrightarrow$ $\exists$ a finite étale cover $u : U \to S$ such that $u^{*}(M, \nabla) \cong (\mathcal{O}_U^r, d)$ the trivial connection

If (M, ∇) on S has regular singularities, this is equivalent to saying that the monodromy representation of (M, ∇) is finite.

Characteristic p

Let T be some base scheme of positive characteristic and let $S \to T$ be a smooth T-scheme.

Let $MIC(S/T)$ be the category of pairs $(\mathcal{E}, \nabla)$ such that $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_S$ module and $\nabla : \mathcal{E} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$ an integrable connection, meaning that

$$\nabla(ge) = g\nabla(e) + dg \otimes e$$

for local sections $g \in \mathcal{O}_S, e \in \mathcal{E}$ and $\nabla^2 = 0$, where

$$\begin{aligned} \nabla : \Omega^i_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} &\to \Omega^{i+1}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \\ \omega \otimes e &\mapsto dw \otimes e + (-1)^i \omega \wedge \nabla(e). \end{aligned}$$

What is now the correct analogue in positive characteristic of a differential equation having a full set of solutions? In characteristic zero, we have that E^{∇} is a sheaf of finite dimensional $\mathbb{C}$-vector spaces. But in positive characteristic, this is not the case. Consider for example the connection $d : k[z] \to k[z]$. Then the kernel is $k[z^p]$, which is not a finit dimensional k-vector space. But it is obviously a finitely generated $k[z^p]$-module. In this case, the analogue of $\mathbb{C}$ here is $k[z^p]$. In general, we need the Frobenius twist and we will replace $\mathbb{C}$ with $\mathcal{O}_{S^{(p)}}$.

Definition 1. Let $f : S \to T$ be a smooth morphism of schemes of positive characteristic p. Let

$$F_{\text{abs}} : T \to T = \begin{cases} \text{id} : |T| \to |T| \\ (-)^p : \mathcal{O}_T \to \mathcal{O}_T \end{cases}$$

be the absolute Frobenius morphism. Then the relative Frobenius morphism $F : S \to S^{(p)}$ is defined the following:

$$\begin{tikzcd}
S \arrow[dr, "F"] \arrow[drr, "F_{\text{abs}}"] \arrow[ddr] & & \\
& S^{(p)} = S \times_{T,F_{\text{abs}}} T \arrow[r] \arrow[d] & S \arrow[d] \\
& T \arrow[r, "F_{\text{abs}}"] & T
\end{tikzcd}$$

For a connection in positive characteristic $(\mathcal{E}, \nabla)$, the horizontal sections $\mathcal{E}^{\nabla}$ are a quasi coherent sheaf on $S^{(p)}$. This is because the Frobenius induces an isomorphism

$$F^* : \mathcal{O}_{S^{(p)}} \to F_* \ker(d : \mathcal{O}_S \to \Omega^1_{X/S})$$

We say that $(\mathcal{E}, \nabla)$ *has a full set of solutions*, if $F^*\mathcal{E}^{\nabla} = \mathcal{E}^{\nabla} \otimes_{\mathcal{O}_{S^{(p)}}} \mathcal{O}_S \cong \mathcal{E}$.

We want to reformulate the property of having a full set of solutions, using the p-curvature.

The p-curvature

Note that for any $D \in \mathcal{D}er(S/T) = \mathcal{D}er_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_S) = \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_{S/T}, \mathcal{O}_S)$ a local section, we get a map $\nabla(D) \in \mathcal{E}nd_T(\mathcal{E})$

$$\nabla(D) : \mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes 1} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \cong \mathcal{E},$$

so ∇ induces a map

$$\nabla : \mathcal{D}er(S/T) \to \mathcal{E}nd_T(\mathcal{E}).$$

Now on both sides we have a p-structure: if D is a derivation, we have the Leibniz rule

$$D^n(gh) = \sum_{i=0}^{n} \binom{n}{i} D^i(g) D^{n-i}(h),$$

so since we are in characteristic p, for $n = p$ we get

$$D^p(gh) = D^p(g)h + gD^p(h),$$

which shows that D^p is again a derivation. We obtain on $\mathcal{D}er(S/T)$ the p-structure

$$(-)^p : \mathcal{D}er(S/T) \to \mathcal{D}er(S/T), \quad D \mapsto D^p.$$

Also on $\mathcal{E}nd_T(\mathcal{E})$, we have a p-structure of just iterating an endomorphism p times,

$$(-)^p : \mathcal{E}nd_T(\mathcal{E}) \to \mathcal{E}nd_T(\mathcal{E}), \quad G \mapsto G^p.$$

This raises the question: (When) is ∇ compatible with the p-structures, i.e., $\nabla(D^p) = (\nabla(D))^p$?

Definition 2. For $(\mathcal{E}, \nabla) \in MIC(S/T)$, we define the *p-curvature*

$$\begin{aligned} \psi : \mathcal{D}er(S/T) &\to \mathcal{E}nd_T(\mathcal{E}) \\ D &\mapsto \psi(D) := (\nabla(D))^p - \nabla(D^p) \end{aligned}$$

Remark 3. The p-curvature is even a map $\psi : \mathcal{D}er(S/T) \to \mathcal{E}nd_S(\mathcal{E})$, i.e., $\psi(D)$ is $\mathcal{O}_S$-linear: Let $g \in \mathcal{O}_s$ and $e \in \mathcal{E}$ be local sections. Since ∇ is a connection, we get the generalized Leibniz rule

$$(\nabla(D))^n(ge) = \sum_{i=0}^{n} \binom{n}{i} D^i(g)(\nabla(g))^{n-i}(e),$$

and therefore

$$(\nabla(D))^p(ge) = D^p(g)e + g(\nabla(D))^p(e).$$

But also because ∇ is a connection, we have

$$\nabla(D^p)(ge) = D^p(g)e + g\nabla(D^p)(e),$$

so by substracting these two equalities, we get

$$\psi(D)(ge) = g\psi(D)(e).$$

The p-curvature gives us now a criterion to check, if a connections has a full set of solutions.

Theorem 4 (Cartier)**.** *Let $f : S \to T$ be a smooth morphism in positive characteristic p. There is an equivalence of categories between the category of quasi coherent sheaves on $S^{(p)}$ and the full subcategory of $MIC(S/T)a$, consisting of objects $(\mathcal{E}, \nabla)$, whoose p-curvature is zero.*

Proof. We have the following functors: Let $\mathcal{F}$ be a quasi coherent sheaf on $S^{(p)}$. Then on $F^*\mathcal{F} = F^{-1}\mathcal{F} \otimes_{F^{-1}\mathcal{O}_{S^{(p)}}} \mathcal{O}_S$, we have the flat connection ∇_{can}, given by

$$\begin{aligned} \nabla_{\text{can}} : F^*\mathcal{F} &\to F^*\mathcal{F} \otimes_{\mathcal{O}_S} \Omega^1_{S/T} \\ f \otimes s &\mapsto (f \otimes 1) \otimes ds \end{aligned}$$

where $s \in \mathcal{O}_s$ and $f \in F^{-1}\mathcal{F}$ local sections. This connection satisfies $(F^*\mathcal{F})^{\nabla_{\text{can}}} \cong \mathcal{F}$ and has vanishing p-curvature. This defines the functor $\mathcal{F} \mapsto (F^*\mathcal{F}, \nabla_{\text{can}})$.

In the other direction, we note that for $(\mathcal{E}, \nabla)$ of p-curvature zero, $\mathcal{E}^\nabla$ is a quasi coherent sheaf on $S^{(p)}$. This is because the Frobenius induces an isomorphism

$$F^* : \mathcal{O}_{S^{(p)}} \to F_* \ker(d : \mathcal{O}_S \to \Omega^1_{X/S})$$

The functor in the other direction is $(\mathcal{E}, \nabla) \mapsto \mathcal{E}^\nabla$.

The only thing left to show is that for a connection $(\mathcal{E}, \nabla)$ with p-curvature zero, the natural map

$$F^*\mathcal{E}^\nabla \to \mathcal{E}$$

For the precise proof, we only consider the case $T = \mathrm{Spec}(k)$ and $S = \mathrm{Spec}(k[z])$. Let ∂_z given by $f \mapsto f'$ be the derivative with respect to z. Then consider the map

$$P = \sum_{j=0}^{p-1} \frac{(-z)^j}{j!} \nabla(\partial_z),$$

which is a $k[z^p]$-linear endomorphism of $k[z]$. The image of P are flat sections. For this note that for $\nabla P = 0$ it suffices $\nabla(\partial_z)P = 0$. We check for $e \in \mathcal{E}$

$$\begin{aligned}
\nabla(\partial_z)(Pe) &= \nabla(\partial_z)(\sum_{j=0}^{p-1} \frac{(-z)^j}{j!} \nabla(\partial_z)e) = \sum_{j=1}^{p-1} \frac{1}{j!} \nabla(\partial_z)((-z)^j \nabla(\partial_z)(e)) \\
&= \sum_{j=0}^{p-1} \frac{1}{j!} (-j(-z)^{j-1} \nabla(\partial_z)(e) + (-z)^j \nabla(\partial_z)^{j+1}(e)) \\
&= \sum_{j=1}^{p-1} (-\frac{(-z)^{j-1}}{(j-1)!} \nabla(\partial_z)^j(e) + (\frac{(-z)^j}{j!} \nabla(\partial_z)^{j+1}(e)) + \nabla(\partial_z)(e) = \nabla(\partial_z)^p(e) = 0
\end{aligned}$$

because of the vanishing p-curvature. Therefore, we already get

$$P : \mathcal{E} \to \mathcal{E}^\nabla.$$

Also, we want to show now that P produces enough horizontal sections. For this, one checks (similar as before) that

$$T(e) := \sum_{k=0}^{p-1} \frac{z^k}{k!} P(\nabla(\partial_z)^k(e)) = e,$$

so T defines an inverse.

In the general case, one uses that since the question is local on S, we can assume that S is affine and étale over $\mathbb{A}^r_T$, so $\Omega^1_{S/T}$ is freely generated by sections $\{ds_1, ..., ds_r\}$. Then

$$P := \sum_\omega \prod_{i=1}^r \left(\frac{(-s_i)^{\omega_i}}{\omega_i!} \prod_{i=1}^r \nabla(\partial_{s_i})^{\omega_i} \right),$$

where the sum is taken over all r-tuples $(\omega_1, ..., \omega_r)$ of integers such that $0 \leq \omega_i \leq p-1$. Then,

$$T(e) := \sum_\omega \prod_{i=1}^r \frac{s_i^{\omega_i}}{\omega_i!} P \prod_{i=1}^r \nabla(\partial_{s_i})^{\omega_i}(e)$$

and one can check that this defines an inverse. □

This shows, that an algebraic differential equation in positive characteristic has a full set of solutions, if and only if the p-curvature vanishes.

We also have the following properties:

Proposition 5. *Let $g \in \mathcal{O}_S, D, D' \in \mathcal{D}er(S/T)$ local sections.*

(1) The map

$$\psi : \mathcal{D}er(S/T) \to \mathcal{E}nd_S(\mathcal{E})$$

is p-linear, i.e., additive and $\psi(gD) = g^p \psi(D)$,

(2) $\nabla(D), \nabla(D^p), \psi(D)$ mutually commute,

(3) $\psi(D)$ and $\psi(D')$ commute,

(4) $\psi(D)$ and $\nabla(D')$ commute.

Now we can state the Grothendieck Katz conjecture precisely.

The Grothendieck Katz conjecture

Let $S_{\mathbb{C}}$ ba a smooth connected quasi-projective $\mathbb{C}$-scheme and let $(M, \nabla)_{\mathbb{C}}$ be an integrable connection on $S_{\mathbb{C}}$. Then we can find a finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$ and a smooth R-scheme S with an integrable connection (M, ∇) on S, which induces $(M, \nabla)_{\mathbb{C}}$ on $U_{\mathbb{C}}$.

For a maximal ideal $\mathfrak{p} \subset R$, $R/\mathfrak{p} \cong \mathbb{F}_q$ is a finite field of characteristic p. So modulo $\mathfrak{p}$, we get a connection $(M/\mathfrak{p}M, \nabla)$ on $S_{\mathfrak{p}}$ in positive characteristic. We say that $S_{\mathbb{C}}$ has *p-curvature zero for almost all p*, if the p-curvature of $(M/\mathfrak{p}M, \nabla)$ vanishes for all but finitely many p. This property does not depend on the choice of the spreading out.

Conjecture 6 (Grothendieck-Katz conjecture)**.** *$(M, \nabla)_{\mathbb{C}}$ has a full set of algebraic solutions if and only if it has p-curvature zero for almost all p.*

The interesting part of of the conjecture is the „if“-part, because if $(M, \nabla)_{\mathbb{C}}$ has a full set of algebraic solutions, it becomes trivial after an étale covering. But the p-curvature of a connection is zero if and only if the p-curvature of an étale covering is zero.

Theorem 13.0 of (Katz: Nilpotent connection and the monodromy theorem) implies that if the p-curvature is zero for almost all p, then the connection $(M, \nabla)_{\mathbb{C}}$ has only regular singular points. Therefore, for it to have a full set of algebraic solutions, it suffices that the monodromy group is finite.

Conjecture 7. *Suppose that the p-curvature is zero for almost all p. Then $(M, \nabla)_{\mathbb{C}}$ has a finite monodromy group.*

Already the rank one case is interesting: Let $(\mathcal{O}_S, \nabla)$ be an integrable rank one equation, given by $\nabla(f) = df + f\nabla(1)$ with a closed form $\omega := \nabla(1)$. If we reduce this equation modulo a maximal ideal in R, then it has a solution, if and only if it is logarithmic, i.e., $\omega = df/f$ for some unit f. On the other hand, the equation admits a solution on a finite étale covering of $S_{\mathbb{C}}$, if and only if $n\omega$ is logarithmic for some $n \geq 1$.

(If $n\omega = dg/g$, then $f = g^{-1/n}$ is an algebraic solution. On the other hand if $\omega = df/f$ for an algebraic function of degree n ($u : U \to S_{\mathbb{C}}$ finite étale of degree n and $f \in \mathcal{O}_U$ solution), then for $g := \mathrm{Norm}(1/f)$, we have that $n\omega = \mathrm{trace}(\omega) = \mathrm{trace}(-df/f) = dg/g$, where $\mathrm{trace} : \Omega^1_{U/\mathbb{C}} \to \Omega^1_{S_{\mathbb{C}}/\mathbb{C}}$, which satisfies $\mathrm{trace}(\frac{df}{f}) = \frac{d\mathrm{Norm}(f)}{\mathrm{Norm}(f)}$ for any $f \in \mathcal{O}_U^{\times}$).

So a special case of the Grothendieck Katz conjecture is the following

Conjecture 8. *Let $\omega \in H^0(S, \Omega^1_{S/R})$ be a closed form. Then ω is logarithmic modulo $\mathfrak{p}$ for almost all maximal ideals in $\mathfrak{p} \subset R$ if and only if a multiple $n\omega$ for $n \geq 1$ is logarithmic on $S_{\mathbb{C}}$*

The question of Grothendieck first arised from the following question: Let

$$\frac{d}{dz}f = A(z)f$$

be a system of linear differential equations, where $A(z) \in \mathrm{Mat}_{n\times n}(K(z))$, where K is a number field. For almost all prime ideals $\mathfrak{p}$ of the ring of integers of K is makes sense to reduce this equation modulo $\mathfrak{p}$, and one gets a differential equation over $\mathbb{F}_q(z)$. Then the Grothendieck Katz conjecture sais:

Conjecture 9. *The differential equavion above has a full set of algebraic solutions, of and only if the reduced equation modulo $\mathfrak{p}$ has a full set of solutions (i.e., n solutions in $\mathbb{F}_q(z)^n$, which are linearly independent over $\mathbb{F}_q(z)$).*

Reduction to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

The Grothendieck Katz conjecture is equivalent to the following special case:

Let $S_{\mathbb{C}} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $M = \mathcal{O}^n_{S_{\mathbb{C}}}$ and consider the differential equation

$$\nabla(f) = df - (\frac{1}{t}B + \frac{1}{t-1}C)f dz,$$

where $B, C \in \mathrm{Mat}_{r\times r}(\overline{\mathbb{Q}})$. We can spread this out to the ring of integers R of $\mathbb{Q}(b_{ij}, c_{ij})$. Then for almost all maximal ideals $\mathfrak{p} \subset R$, we can reduce the equation modulo $\mathfrak{p}$.

We want to show that the Grothendieck Katz conjecture is equivalent to the Grothendieck Katz conjecture for this specific situation. We sketch, how one can reduce the general conjecture to this case. So let $(M, \nabla)_{\mathbb{C}}$ be an integrable connection on a smooth connected quasi-projective $\mathbb{C}$-scheme $S_{\mathbb{C}}$ and consider a spreading out (M, ∇) on S over R. Theorem 13.0 in (Katz: Nilpotent connections and the monodromy theorem) implies the following:

Theorem 10. *If the $(M, \nabla)_{\mathbb{C}}$ has vanishing p-curvature for almost all p, then $(M, \nabla)_{\mathbb{C}}$ is regular singular and has finite monodromies at infinity.*

Claim 11. *The Grothendieck Katz conjecture is equivalent to the Grothendieck Katz conjecture, where $S_{\mathbb{C}}$ is a smooth projective curve.*

Proof. Since $S_{\mathbb{C}}$ has only regular singularities, we can choose a compactification $\overline{S_{\mathbb{C}}}$ of $S_{\mathbb{C}}$, which is projective, and such that the complement $D = \overline{S_{\mathbb{C}}} \setminus S_{\mathbb{C}}$ is a simple normal crossing divisor. The connection $(M, \nabla)_{\mathbb{C}}$ extends to a locally free sheaf of finite rank $\overline{M}$ on $\overline{S_{\mathbb{C}}}$ and a map

$$\overline{\nabla} : \overline{M} \to \Omega^1_{\overline{S_{\mathbb{C}}}/\mathbb{C}}(\log D) \otimes \overline{M},$$

extending ∇ and also satisfying a Leibniz rule. Now since $S_{\mathbb{C}}$ has only finite monodromy at infinity, we can reach after a finite cover $u' : U' \to \overline{S_{\mathbb{C}}}$ that we get a connection

$$\nabla : \overline{M}_{U'} \to \Omega^1_{U'/\mathbb{C}} \otimes \overline{M}_{U'},$$

which means that we can remove the log-poles.

(For example, consider $S_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[z, \frac{1}{z}])$, $M = \mathbb{C}[z, \frac{1}{z}]$ and the differential equation

$$\nabla(f) = df - bf\frac{dz}{z},$$

with $b \in \overline{\mathbb{Q}}$. Finite monodromy at infinity means that the image of all loops around a point at infinity under the monodromy representation is finite. In this case, this means (see intruduction talk) that $b \in \mathbb{Q}$. This is because a the loops around zero induce the representation

$$\mathbb{Z} \ni l \mapsto e^{2\pi ibl},$$

which is finite, if and only if $b \in \mathbb{Q}$. But also note, what a base change of the basis of $\mathcal{O}_{\overline{S_{\mathbb{C}}}}$ does: Let m, m' be two elements, generating $\mathcal{O}_{\overline{S_{\mathbb{C}}}}$. Let m be the standard basis, in which the extention of the connection is given by

$$\overline{\nabla} : \mathcal{O}_{\overline{S_{\mathbb{C}}}} \to \Omega^1_{\overline{S_{\mathbb{C}}}/\mathbb{C}}(\log(z)) \otimes \mathcal{O}_{\overline{S_{\mathbb{C}}}},$$
$$m \mapsto b\frac{dz}{z} \otimes m.$$

By replacing $\overline{S_{\mathbb{C}}}$ by a finite extention, we can assume that $z^{-b} \in \mathcal{O}^{\times}_{\overline{S_{\mathbb{C}}}}$ and we write $m' = z^{-b}m$. Then we have

$$\overline{\nabla}(z^{-b}m) = d(z^{-b}) \otimes m + z^{-b}(b\frac{dz}{z} \otimes m) = z^b d(z^{-b}) \otimes m' + (b\frac{dz}{z}) \otimes m'$$
$$= (z^b \cdot (-bz^{-b-1}dz)) \otimes m' + (b\frac{dz}{z}) \otimes m' = 0,$$

so after the base change to m', the log pole vanishes. This showes that after a finite extention we really have a connection. end of example)

Since for having a full set of algebraic solutions it suffices that there is a trivializing finite cover (instead of a finite étale cover), we can reduce to the case that $S_{\mathbb{C}}$ is projective.

Now we can use the Lefschetz hyperplane theorem, which tells us that there is a smooth projective connected curve $C \subset S_{\mathbb{C}}$, such that the induced map

$$\pi_1(C^{an}) \to \pi_1((S_{\mathbb{C}})^{an})$$

is surjective. This reduces the Grothendieck Katz conjecture to the case of a smooth projective connected curve. □

Claim 12. *The Grothendieck Katz conjecture is equivalent to the Grothendieck Katz conjecture for smooth projective connected curves defined over a number field.*

Proof. For $(M, \nabla)_{\mathbb{C}}$ on $S_{\mathbb{C}}$, we get the spreading out (M, ∇) on S over R. If we consider the base change $R_{\mathbb{Q}} = R \otimes_{\mathbb{Z}} \mathbb{Q}$, then we get a connection $(M, \nabla)_{\mathbb{Q}}$ on $S_{\mathbb{Q}}$. By a Corollaire 7.1.3 of (Yves Adreé: Sur la conjecture des p-courbures de Grothendieck–Katz et un problème de Dwork), the generic fiber satisfies the Grothendieck Katz conjecture if and only if all fibers of closed point of a dense open set of $\mathrm{Spec}(R_{\mathbb{Q}})$ satisfy the Grothendieck Katz conjecture. □

Claim 13. *The Grothendieck Katz conjecture is equivalent to the situation described above.*

Proof. Let S be defined over a numberfield. Since $\pi_1(U) \twoheadrightarrow \pi_1(S)$ for a nonempty Zariski open, we can assume that $S_{\mathbb{C}}$ is an affine curve. Then, by a theorem of Belyi, there is a finite étale cover $\pi : S \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ (since we already reduced to the case of a curve over a number field), so we are reduced to the case $S_{\mathbb{C}} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Then one can further reduce it to the specific equation given above (see for example Remarque 7.1.4 in Yves André: Sur la conjecture des p-courbures de Grothendieck–Katz et un problème de Dwork). □

Example

Consider the differential equation

$$\nabla(f) = df - bf\frac{dz}{z} = 0$$

on $\mathbb{C}[z, \frac{1}{z}]$ from above, where $b \in \overline{\mathbb{Q}}$. We already saw in the introduction talk that there is an algebraic solution if and only if $b \in \mathbb{Q}$.

On the other hand, we can reduce this equation modulo almost all prime ideals of $\mathbb{Q}(b)$ (more specific: we can reduce it for all primes $\mathfrak{p}$ in $\mathbb{Q}(b)$, where b is integral with respect to the induced non-Archimedian norm). Since we only have one variable, the p-curvature ψ_p is zero, if and only if $\psi_p(\partial_z) = 0$, where $\partial_z(f) = f'$, this corresponds to the morphism $\phi : \Omega^1_{\mathbb{C}[z,\frac{1}{z}]/\mathbb{C}} \to \mathbb{C}[z, \frac{1}{z}],\ dz \mapsto 1$. Since $\partial_z^p = 0$ this is equivalent to

$$\nabla(\partial_z)^p = 0.$$

For $f \in \mathbb{C}[z, \frac{1}{z}]$,

$$\nabla(\partial_z)(f) = \phi(\nabla(f)) = \phi(df - bf\frac{dz}{z}) = f' - \frac{b}{z}f = (\frac{d}{dz} - \frac{b}{z})f,$$

therefore for the p-curvature to vanish, it is equivalent, that

$$0 = (\frac{d}{dz} - \frac{b}{z})^p z^n = (n-b)(n-b-1)...(n-b-p+1)z^{n-p} \equiv 0 \mod p$$

for almost all maximal ideals $\mathfrak{p}$ of $\mathbb{Q}(b)$. We see that the p-curvature is zero for almost all p, if and only of for almost all p there is an integer $r \in \mathbb{Z}$ such that $b \equiv r \bmod p$. We want to show that this is equivalent to $b \in \mathbb{Q}$.

If $b \in \mathbb{Q}$, then this is definitely fulfilled: write $b = \frac{m}{k}$ with $(m, k) = 1$. Then for any p prime, which does not divide k, b defines an element in $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$. So $b \equiv r \bmod p$ for some $r \in \mathbb{Z}$.

On the other hand assume that for almost all p there is an integer $r \in \mathbb{Z}$ such that $b \equiv r \bmod p$. By replacing b by some multiple, we can assume that it is integral over $\mathbb{Z}$. By the Chebotarev density theorem, it suffices to show that almost all p split completely in $\mathbb{Q}(b)$. To show this, let L be the Galois hull of $\mathbb{Q}(b)/\mathbb{Q}$. Then it suffices to show that almost all p split completely in L. Let $b_1, ..., b_n$ be the Galois conjugates of b. Then $L = \mathbb{Q}(b_1, ..., b_n)$. Let $B = \mathcal{O}_L$ and $\mathbb{Z}[b_1, ..., b_n]$. Since $\mathrm{Frac}(C) = L$ is Galois over $\mathbb{Q}$, the map $\mathrm{Spec}(C) \to \mathrm{Spec}(\mathbb{Z})$ is étale over a dense open subset $U \subset \mathrm{Spec}(\mathbb{Z})$. So for any $p \in U$, $C_{(p)}$ is étale over $\mathbb{Z}_{(p)}$. Therefore $C_{(p)}$ is regular and therefore integrally closed. But since B is integrally closed, this implies $C_{(p)} = B_{(p)}$ and $B/pB = C/pC$. Also we have for $p \in U$ the decomposition in primes of B,

$$pB = q_1, ..., q_m,$$

where the q_i are all pairwise different and they all have the same residue field degree

$$f := [B/q_i : \mathbb{F}_p].$$

This implies that

$$C/pC = \prod_{i=1}^m \mathbb{F}_{p^f}$$

To show that p splits completely in L, we have to show that $f = 1$. For this, it suffices to show that there is a nontrivial ring homomorphism $C/pC \to \mathbb{F}_p$. By assumption, we know that $b \equiv r \mod p$, so it lies in the image of $\mathbb{Z}$ in C/pC. So we define the morphism to be

$$C/pC \to \mathbb{F}_p, \quad b_i \mapsto r \mod p.$$

Note also that $b \notin pB$ for all but finitely many p, because we know that the ideal in pB has a finite decomposition into prime ideals. So the defined map is not zero for all but finitely many prime numbers p. This shows that almost all p split completely in L and we conclude that $b \in \mathbb{Q}$.

We see that this specific differential equation satisfies the Grothendieck Katz conjecture.

Another connection, which satisfies the Grothendieck Katz conjecture is the Gauß-Manin connection.

Theorem 14. *The Gauß-Manin connection satisfies the Grothendieck-Katz conjecture.*