Tilting

Today, we want to introduce the tilting functor and see a few properties. In particular, we will show that it is related to the functor of ramified Witt vectors, which we saw last week.

Setting

For this talk we fix:

- $p \in \mathbb{Z}$ prime
- $E/\mathbb{Q}_p$ a finite field extension
- $\mathcal{O}_E = \{x \in E \,|\, |x| \leq 1\} \subset E$ the ring of integers
- $\pi \in E$ a uniformizer, i.e., $(\pi) = \mathfrak{m} = \{x \in E \,|\, |x| < 1\} \subset \mathcal{O}_E$ the maximal ideal
- $\mathbb{F}_q = \mathcal{O}_E/(\pi)$ the residue field

Before we start with the definition of the tilt, we present a lemma, which will be used several times in the following.

Lemma 1. *Let A be an $\mathcal{O}_E$-algebra and $I \subset A$ an ideal such that $\pi \in I$ and let $a, b \in A$ such that $a \equiv b \mod I$. Then we have*

$$a^{q^k} \equiv b^{q^k} \mod I^{k+1}$$

for all $k \geq 0$.

Proof. By induction, it suffices to show that $a \equiv b \mod I^k$ implies $a^q \equiv b^q \mod I^{k+1}$ for $k \geq 1$. For this, we write $b = a + c$ with $c \in I^k$. Then

$$b^q = (a+c)^q = \sum_{i=0}^{q} \binom{q}{i} a^i c^{q-i} \equiv a^q \mod I^{k+1},$$

since $c \in I^k$ and $q \in (\pi) \subset I$. $\square$

The tilting functor

Definition 2. Let A be a π-complete $\mathcal{O}_E$-algebra. Then we define

$$A^\flat := \varprojlim_{x \mapsto x^q} A/\pi = \{(a_0, a_1, \dots,) \in \prod_{\mathbb{N}} A/\pi \,|\, a_{i+1}^q = a_i\}$$

the *tilt* of A.

The tilt can be defined for general $\mathcal{O}_E$-algebras, but in the following we will only consider π-complete ones and therefore only define the tilt for these.

Remark 3. (1) $A^\flat$ is a ring, since $(-)^q : A/\pi \to A/\pi$ is a ring homomorphism

(2) The tilt $A^\flat$ is an $\mathcal{O}_E^\flat = \mathbb{F}_q$-algebra

(3) $A^\flat$ is perfect, i.e., $(-)^q : A^\flat \to A^\flat$ is bijective. Namely, it has the inverse

$$(a_0, a_1, \dots) \mapsto (a_1, a_2, \dots).$$

Therefore, we get a functor

$$(-)^\flat : \{\pi - \text{complete } \mathcal{O}_E - \text{algebras}\} \to \{\text{perfect } \mathbb{F}_q - \text{ algebras}\}$$

Proposition 4. *Let A be a π-complete $\mathcal{O}_E$-algebra, $I \subset A$ an ideal such that $\pi \in I$ and A is I-adically complete. Then the map*

$$\varprojlim_{x \mapsto x^q} A \to (A/I)^\flat, \quad (a_0, a_1, \dots) \mapsto (\bar{a}_0, \bar{a}_1, \dots)$$

is a bijective morphism of multiplicative monoids.

Proof. We want to construct an inverse map. For this, let $x = (x_0, x_1, \dots) \in (A/I)^\flat$ and consider $\tilde{x}_i \in A$ lifts of the x_i.
Claim: $\{\tilde{x}_i^{q^i}\}_{i \geq 0}$ is a Cauchy sequence with respect to the I-adic topology.
To show this, let $j \geq i$. Then $x_j^{q^{i-j}} = x_i$ in A/I, i.e., $\tilde{x}_j^{q^{i-j}} \equiv \tilde{x}_i \mod I$. By Lemma 1,

$$\tilde{x}_j^{q^j} \equiv \tilde{x}_i^{q^i} \mod I^{i+1}$$

and therefore $\{\tilde{x}_i^{q^i}\}_{i \geq 0}$ is a Cauchy sequence. Since A is I-adically complete, the limit of this sequence exists. We define

$$x^\sharp := \lim_{i \to \infty} \tilde{x}_i^{q^i} \in A$$

Claim: $x^\sharp$ is independent of the choice of the $\tilde{x}_i$.
For this, let $\tilde{y}_i \equiv \tilde{x}_i \mod I$. By Lemma 1, this implies $\tilde{y}_i^{q^i} \equiv \tilde{x}_i^{q^i} \mod I^{i+1}$ and therefore

$$\lim_{i \to \infty} \tilde{y}_i^{q^i} = \lim_{i \to \infty} \tilde{x}_i^{q^i}.$$

We get a map

$$(-)^\sharp : (A/I)^\flat \to A,$$

which is well-defined and clearly multiplicative. Now define

$$(A/I)^\flat \to \varprojlim_{x \mapsto x^q} A, \quad x \mapsto (x^\sharp, (x^{1/q})^\sharp, (x^{1/q^2})^\sharp, \dots).$$

This is an inverse: Let $(a_0, a_1, \dots) \in \varprojlim_{x \mapsto x^q} A$

$$(a_0, a_1, \dots) \mapsto \bar{a} = (\bar{a}_0, \bar{a}_1, \dots) \mapsto (\bar{a}^\sharp, (\bar{a}^{1/q})^\sharp, \dots) = (\lim_{i \to \infty} a_i^{q^i}, \lim_{i \to \infty} a_{i+1}^{q^i}, \dots),$$

and since $a_{i+j}^{q^i} = a_j$, we have

$$\lim_{i\to\infty} a_{i+j}^{q^i} = \lim_{i\to\infty} a_j = a_j.$$

On the other hand, for $x = (x_0, x_1, \dots) \in (A/I)^\flat$, then

$$(x_0, x_1, \dots) \mapsto (x^\sharp, (x^{1/q})^\sharp, \dots) = (\lim_{i\to\infty} \tilde{x}_i^{q^i}, \lim_{i\to\infty} \tilde{x}_{i+1}^{q^i}, \dots) \mapsto (\lim_{i\to\infty} x_i^{q^i}, \lim_{i\to\infty} x_{i+1}^{q^i}, \dots) = x$$

by the same argument as before. □

Remark 5. (1) In particular, we get an additive structure on the left hand side by using the additive structure on the tilt: Let $(a_0, a_1, \dots), (b_0, b_1, \dots) \in \varprojlim_{x\mapsto x^q} A$, then

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (\lim_{i\to\infty} (a_i + b_i)^{q^i}, \lim_{i\to\infty} (a_{i+1} + b_{i+1})^{q^i}, \dots)$$

(2) Also, by the proposition, we can conclude that we have

$$A^\flat \cong \varprojlim_{x\mapsto x^q} A$$

as multiplicative monoids (by taking $I = (\pi)$). In particular, we see that for $x = (x_0, x_1, \dots) \in A^\flat$, $(x^{1/q^n})^\sharp$ is a lift of x_n modulo π.

Last week, the ramified Witt vectors were introduced. We now want to show, how they are connected to the tilt.

Proposition 6. *The functor*

$$(-)^\flat : \{\pi\text{-complete } \mathcal{O}_E\text{-algebras}\} \to \{\text{perfect } \mathbb{F}_q\text{-algebras}\}$$

is right adjoint with left adjoint given by the functor $W_{\mathcal{O}_E}(-)$.

Proof. We want to construct the unit and the counit of the adjunction. The unit is given by

$$\begin{aligned} \eta\colon R \to W_{\mathcal{O}_E}(R)^\flat &= \varprojlim_{x\mapsto x^q} W_{\mathcal{O}_E}(R)/\pi = \varprojlim_{x\mapsto x^q} R, \\ r &\mapsto (r, r^{1/q}, r^{1/q^2}, \dots). \end{aligned}$$

Note that this is an isomorphism, which implies in particular that $W_{\mathcal{O}_E}(-)$ is fully faithful.

The counit $\theta : W_{\mathcal{O}_E}(A^\flat) \to A$ (for a π-complete $\mathcal{O}_E$-algebra A) is called Fontaine's map. For the construction, we first note that by definition of the Witt vectors, the map

$$\mathcal{W}_n : W_{\mathcal{O}_E}(A) \to A/\pi^{n+1}, \quad (a_0, a_1, \dots) \mapsto \sum_{i=0}^{n} a_i^{q^{n-i}} \pi^i$$

is a morphism of rings. If we further assume that $a_i \equiv 0 \mod \pi$, then $a_i^{q^{n-i}} \equiv 0 \mod \pi^{n-i+1}$ for all $0 \le i \le n$ by Lemma 1. Therefore,

$$\sum_{i=0}^{n} a_i^{q^{n-i}} \pi^i \equiv 0 \mod \pi^{n+1}.$$

and therefore, $\mathcal{W}_n$ factors over $W_{\mathcal{O}_E}(A/\pi)$ and clearly also over

$$W_{\mathcal{O}_E,n}(A/\pi) = W_{\mathcal{O}_E}(A/\pi)/V^{n+1}(W_{\mathcal{O}_E}(A/\pi)),$$

i.e., we get a well defined map

$$\theta_n : W_{\mathcal{O}_E,n}(A/\pi) \to A/\pi^{n+1}.$$

We get a diagram

$$\begin{tikzcd} W_{\mathcal{O}_E,n+1} \arrow[r, "\theta_{n+1}"] \arrow[d, "F"] & A/\pi^{n+2} \arrow[d] \\ W_{\mathcal{O}_E,n} \arrow[r, "\theta_n"] & A/\pi^{n+1} \end{tikzcd}$$

where $F((a_0, ..., a_{n+1}) = (a_0^q, ..., a_n^q)$ and map $A/\pi^{n+2} \to A/\pi^{n+1}$ is just the natural projection. The diagram commutes, since

$$\begin{aligned} \theta_n(F(a_0, \dots, a_{n+1})) = \theta_n(a_0^q, \dots, a_n^n) &= \sum_{i=0}^{n} a_i^{n+1-i} \pi^i \\ &\equiv \sum_{i=0}^{n+1} a_i^{n+1-i} \pi^i = \theta_{n+1}(a_0, \dots, a_{n+1}) \mod \pi^{n+1}. \end{aligned}$$

By passing to the limit, we get a map

$$\theta : W_{\mathcal{O}_E}(A^\flat) \cong \varprojlim_{n,F} W_{\mathcal{O}_E,n}(A/\pi) \to \varprojlim_{n} A/\pi^{n+1} \cong A$$

Now we have to check that these maps really give us the desired adjunction. First, let $f \colon W_{\mathcal{O}_E}(R) \to A$ be a morphism of $\mathcal{O}_E$-algebras (R is a perfect $\mathbb{F}_q$-algebra). Then we have

$$\begin{aligned} f^\flat \circ \eta \colon R &\to W_{\mathcal{O}_E}(R)^\flat \to A^\flat, \\ r &\mapsto (r, r^{1/q}, r^{1/q^2}, \dots) \mapsto (\bar{f}(r), \bar{f}(r^{1/q}), \dots) \end{aligned}$$

where $\bar{f} : R \cong W_{\mathcal{O}_E}(R)/\pi \to A/\pi$ is the map induced by f. We get the map

$$\begin{aligned} W_{\mathcal{O}_E}(f^\flat \circ \eta) \colon W_{\mathcal{O}_E}(R) &\to W_{\mathcal{O}_E}(A^\flat) \\ x = (x_0, x_1, \dots) &\mapsto ((\bar{f}(x_0), \bar{f}(x_0^{1/q}), \dots), (\bar{f}(x_1), \bar{f}(x_1^{1/q}), \dots), \dots) \\ &= [(\bar{f}(x_0), \bar{f}(x_0^{1/q}), \dots)] + [(\bar{f}(x_1^{1/q}), \bar{f}(x_1^{1/q^2}), \dots)] \cdot \pi + \dots \end{aligned}$$

and therefore

$$\theta \circ W_{\mathcal{O}_E}(f^\flat \circ \eta)(x) = \sum_{i=0}^{\infty} f([x_i^{1/q^i}])\pi^i = f(\sum_{i=0}^{\infty}[x_i^{1/q^i}]\pi^i) = f(x).$$

For the other direction, let $g\colon R \to A^\flat$ be a morphism of $\mathbb{F}_q$-algebras. For $r \in R$, we denote $g(r) = (g(r)_0, g(r)_1, \dots) \in A^\flat$. We have

$$\begin{aligned} \theta \circ W_{\mathcal{O}_E}(g)\colon W_{\mathcal{O}_E}(R) &\to W_{\mathcal{O}_E}(A^\flat) \to A \\ (r_0, r_1, \dots) &\mapsto (g(r_0), g(r_1), \dots) = \sum_{i=0}^{\infty}[g(r_i)^{1/q^i}]\pi^i \mapsto \sum_{i=0}^{\infty}(g(r_i)^{1/q^i})^\sharp \pi^i \end{aligned}$$

and therefore

$$\begin{aligned} (\theta \circ W_{\mathcal{O}_E}(g))^\flat \colon W_{\mathcal{O}_E}(R)^\flat &= \varprojlim_{x \mapsto x^q} W_{\mathcal{O}_E}(R)/\pi \to A^\flat \\ ([x_0], [x_1], \dots) &\mapsto (g(x_0)_0, g(x_1)_0, \dots) \end{aligned}$$

because $[x_i] = (x_i, 0, \dots) \in W_{\mathcal{O}_E}(R)$ and $g(x_i)_0$ is a lift of $g(x_i)^\sharp$ modulo π. Now this gives us

$$\begin{aligned} (\theta \circ W_{\mathcal{O}_E}(g))^\flat \circ \eta(r) &= (\theta \circ W_{\mathcal{O}_E}(g))^\flat([r], [r^{1/q}], [r^{1/q^2}], \dots) = (g(r)_0, r(r^{1/q})_0, g(g^{1/q^2})_0, \dots) \\ &= g(r)_0, g(r)_0^{1/q}, g(r)_0^{1/q^2}, \dots) = g(r)_0, g(r)_1, g(r)_2, \dots) = g(r). \end{aligned}$$

Note that for the computations we use the explicit description of the map θ from Lemma 7. $\square$

Lemma 7. *Let A be a π-complete $\mathcal{O}_E$-algebra. Then θ is given by*

$$\theta\left(\sum_{i=0}^{\infty}[a_i]\pi^i\right) = \sum_{i=0}^{\infty} a_i^\sharp \pi^i.$$

Proof. First, we note that every element in $W_{\mathcal{O}_E}(A^\flat)$ can be written as $\sum_{i=0}^{\infty}[a_i]\pi^i$. Now we can check the statement of the Lemma on a finite level of the limit. For this, note that an element $a = (a_0, a_1, \dots) \in W_{\mathcal{O}_E}(A^\flat) = \varprojlim_{n,F} W_{\mathcal{O}_E,n}(A/\pi)$ with $a_i = (a_{i,0}, a_{i,1}, \dots) \in A^\flat$, is corresponding to the element $(a_{0,n}, a_{1,n}, \dots, a_{n,n}) \in W_{\mathcal{O}_E,n}(A/\pi)$ on the finite level n. Now we have

$$\sum_{i=0}^{\infty}[a_i]\pi^i = (a_0, a_1^q, a_2^{q^2}, \dots) \in W_{\mathcal{O}_E}(A^\flat),$$

and therefore, it corresponds to the element $(a_{0,n}, a_{1,n}^q, a_{2,n}^{q^2}, \dots, a_{n,n}^{q^n}) = (a_{0,n}, a_{1,n-1}, a_{2,n-2}, \dots, a_{n,0})$, and

$$\theta_n(a_{0,n}, a_{1,n-1}, a_{2,n-2}, \dots, a_{0,n}) = \sum_{i=0}^{n} a_{i,n-i}^{q^{n-i}}\pi^i$$

But now $(a_i^{1/q^{n-i}})^\sharp$ is a lift of $a_{i,n-i}$ modulo π and therefore

$$\sum_{i=0}^{n}((a_i^{1/q^{n-i}})^\sharp)^{q^{n-i}}\pi^i = \sum_{i=0}^{n} a_i^\sharp \pi^i \equiv \sum_{i=0}^{\infty} a_i^\sharp \pi^i \mod \pi^{n+1}$$

is a lift of $\sum_{i=0}^n a_{i,n-i}^{q^{n-i}}\pi^i$ modulo π^{n+1}. $\square$

Recall 8. (1) A *perfect prism* is a pair $(W_{\mathcal{O}_E}(R), I)$, where R is a perfect $\mathbb{F}_q$-algebra and $I \subset W_{\mathcal{O}_E}(R)$ is an ideal, generated by an element $d \in W_{\mathcal{O}_E}(R)$, such that $\frac{F(d)-d^q}{\pi} \in W_{\mathcal{O}_E}(R)^\times$ (i.e., d is distinguished) and $W_{\mathcal{O}_E}(R)$ is I-adically complete.

(2) An $\mathcal{O}_E$-algebra A is *perfectoid*, if $A \cong W_{\mathcal{O}_E}(R)/I$ for a perfect prism $(W_{\mathcal{O}_E}(R), I)$.

Proposition 9 (Tilting equivalence)**.** *Let A be a perfectoid $\mathcal{O}_E$-algebra. Then the functor*

$$\{\textit{perfectoid } A\textit{-algebras}\} \to \{\textit{perfect } A^\flat\textit{-algebras}\}$$
$$B \mapsto B^\flat$$

is fully faithful with essential image all perfect $A^\flat$-algebras S, such that $W_{\mathcal{O}_E}(S)$ is I-adically complete when writing $A \cong W_{\mathcal{O}_E}(A^\flat)/I$.

Proof. Note first that for any perfect $\mathbb{F}_q$-algebra R, there is an equivalence

$$\{\text{untilts } (A,\iota) \text{ of } R \text{ over } \mathcal{O}_E\} \cong \{I \subset W_{\mathcal{O}_E}(R) \text{ such that } (W_{\mathcal{O}_E}(R), I) \text{ is a prism}\}.$$

Also one checks that if $(W_{\mathcal{O}_E}(R), I) \to (W_{\mathcal{O}_E}(S), J)$ is a morphism of prisms, then necessarily $J = IW_{\mathcal{O}_E}(S)$.

We also use in this proof that for any perfectoid $A \cong W_{\mathcal{O}_E}(R)/I$, we have

$$A^\flat \cong (W_{\mathcal{O}_E}(R)/I)^\flat \cong R$$

by using Proposition 4.

With this info, it is easy to first check the statement about the essential image. First note that if we start with a perfectoid A-algebra B, then $B \cong W_{\mathcal{O}_E}(B^\flat)/J$ for $J \subset W_{\mathcal{O}_E}(B^\flat)$ ideal such that $(W_{\mathcal{O}_E}(B^\flat), J)$ is a prism. But since $(W_{\mathcal{O}_E}(A^\flat), I) \to (W_{\mathcal{O}_E}(B^\flat), J)$ is a morphism of prisms, we know that $J = IW_{\mathcal{O}_E}(B^\flat)$ and therefore $W_{\mathcal{O}_E}(B^\flat)$ is I-adically complete. If we have a perfect $A^\flat$-algebra such that $W_{\mathcal{O}_E}(S)$ is I-adically complete, then $(W_{\mathcal{O}_E}(S), IW_{\mathcal{O}_E}(S))$ is a perfect prism and therefore $W_{\mathcal{O}_E}(S)/IW_{\mathcal{O}_E}(S)$ is an untilt of S, which shows that S lies in the essential image.

It is now also clear that we can define an inverse functor from the essential image of $(-)^\flat$ in the other direction:

$$S \mapsto W_{\mathcal{O}_E}(S)/IW_{\mathcal{O}_E}(S).$$

Then clearly

$$W_{\mathcal{O}_E}(B^\flat)/IW_{\mathcal{O}_E}(B^\flat) \cong B,$$

for B perfectoid (by the above argumentation). If on the other hand we start with a $A^\flat$-algebra S in the essential image, then

$$(W_{\mathcal{O}_E}(S)/IW_{\mathcal{O}_E}(S))^\flat \cong S.$$

$\square$

Important example of a perfectoid $\mathcal{O}_E$-algebra

For the rest of the talk, we want to consider one important (maybe the most important) example of a perfectoid $\mathcal{O}_E$-algebra.

Proposition 10. *Let C be an algebraically closed, non-archimedean extention of E (note that by definition, C is a completely valued field) with valuation*

$$\nu\colon C \to \mathbb{R} \cup \{\infty\}.$$

Then the ring of integers

$$\mathcal{O}_C := \{x \in C \mid \nu(x) \geq 0\}$$

is a perfectoid $\mathcal{O}_E$-algebra.

Proof. We need to show that $\mathcal{O}_C \cong W_{\mathcal{O}_E}(R)/I$ for a perfect prism $(W_{\mathcal{O}_E}(R), I)$. Actually, in this situation, there is an obvious choice for R, namely $\mathcal{O}_C^\flat$. Therefore, what we will show is that $\theta : W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \to \mathcal{O}_C$ is surjective and $\ker\theta$ is generated by a distinguished element.

Claim: $\theta\colon W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \to \mathcal{O}_C$ is surjective.

Let $a \in \mathcal{O}_C$. Since C is algebraically closed, we can find elements $b_1, b_2, \dots \in \mathcal{O}_C$ such that $a_0 = (a, b_1, b_2, \dots)$ defines an element in $\varprojlim_{x\mapsto x^q} \mathcal{O}_C \cong \mathcal{O}_C^\flat$. Then $a_0^\sharp \in \mathcal{O}_C$ is a lift of a modulo π. Therefore, $a - a_0^\sharp = \pi a'$ and we can do the same thing for a'. Doing this inductively, we get elements $a_i \in \mathcal{O}_C^\flat$ such that $\sum_{i=0}^n a_i^\sharp \pi^i$ is a lift of a modulo π^{i+1} and therefore $a = \sum_{i=0}^\infty a_i^\sharp \pi^i$.

Let $\pi^{1/q^n} \in \mathcal{O}_C$ be a compatible system of q^n-th roots of π. These exist since C is algebraically closed. Let

$$\pi^\flat := (\pi, \pi^{1/q}, \dots) \in \varprojlim_{x\mapsto x^q} \mathcal{O}_C \cong O_C^\flat.$$

Claim: $\mathcal{O}_C^\flat/\pi^\flat \cong \mathcal{O}_C/\pi$ via the $\sharp$-map.

For this, consider the $\sharp$-map

$$(-)^\sharp : \varprojlim_{x\mapsto x^q} \mathcal{O}_C \cong \mathcal{O}_C^\flat \to \mathcal{O}_C, \quad (y_0, y_1, \dots) \mapsto y_0$$

Let $y = (y_0, y_1, \dots) \in \varprojlim_{x\mapsto x^q} \mathcal{O}_C \cong \mathcal{O}_C^\flat$. Then $\pi^\flat | y$ if and only if $\pi^{1/q^n} | y_n$ for all n. Since we are in a valuation ring, this happens if and only if $\nu(y_n) \geq \nu(\pi^{1/q^n}) = q^{-n}\nu(\pi)$ for all n, i.e., if and only if $\nu(y_0) \geq \nu(\pi)$ (as $y_0 = y_n^{q^n}$, we have $\nu(y_0) = q^n\nu(y_n)$ and therefore $\nu(y_n) = q^{-n}\nu(y_0)$). But again, using that $\mathcal{O}_C$ is a valuation ring, this happens if and only if $y_0 \equiv 0 \mod \pi$. Therefore,

$$\ker(\mathcal{O}_C^\flat \xrightarrow{(-)^\sharp} \mathcal{O}_C \to \mathcal{O}_C/(\pi)) = (\pi^\flat).$$

Since the $\sharp$-map is surjective, we conclude $\mathcal{O}_C^\flat/\pi^\flat \cong \mathcal{O}_C/\pi$.

We now want to show that $\xi := \pi - [\pi^\flat]$ generates $\ker(\theta\colon W_{\mathcal{O}_E}(\mathcal{O}_C^\flat) \to \mathcal{O}_C)$. Since $(W_{\mathcal{O}_E}(\mathcal{O}_C^\flat), \xi)$ is a perfect prism, this will finish the proof (note: $\xi = [r_0] + [r_1]\pi$ with $r_0 = \pi^\flat$ and $r_1 = 1 \in (\mathcal{O}_C^\flat)^\times$ clearly. Therefore ξ is distinguished and since $\mathcal{O}_C$ is π-complete, $\varprojlim_{x \mapsto x^q} \mathcal{O}_C$ is $\pi^\flat$-complete: We have

$$\varprojlim_n((\varprojlim_{x\mapsto x^q} \mathcal{O}_C)/(\pi^\flat)^n) = \varprojlim_{x\mapsto x^q}(\varprojlim_n \mathcal{O}_C/\pi^n) \cong \varprojlim_{x\mapsto x^q} \mathcal{O}_C.)$$

For this, we first note that

$$\theta(\pi - [\pi^\flat]) = \pi - (\pi^\flat)^\sharp = \pi - \pi = 0,$$

so $\xi \in \ker(\theta)$.
It remains to show that ξ generates $\ker(\theta)$. For this, let $x = \sum_{i=0}^\infty [x_i]\pi^i \in \ker(\theta)$. Then

$$0 = \theta(x) = \sum_{i=0}^\infty x_i^\sharp \pi^i$$

and therefore $x_0^\sharp \equiv 0 \mod \pi$. By the above argumentation, this implies that $\pi^\flat | x_0$. Writing $z_0 = \sum_{i=1}^\infty [x_i]\pi^{i-1}$, we get

$$x = [x_0] + \pi z_0 = [x_0] + \pi z_0 - [\pi^\flat]z_0 + [\pi^\flat]z_0 = [x_0] + [\pi^\flat]z_0 + \xi z_0 = [\pi^\flat]x' + \xi z_0,$$

since $\pi^\flat | x_0$ and the Teichmüller lift is multiplicative. But now

$$0 = \theta(x) = \theta([\pi^\flat]x') + \theta(\xi z_0) = \theta([\pi^\flat]x') = \pi\theta(x'),$$

which implies $\theta(x') = 0$ because $\pi \in \mathcal{O}_C$ is a non-zero divisor. We can do the same thing again for x', i.e., write $x' = [\pi^\flat]x'' + \xi z_1$ and so on to get

$$x = \xi z_0 + [\pi^\flat]x' = \xi z_0 + [\pi^\flat](\xi z_1 + [\pi^\flat]x'') = \cdots = \xi \cdot \sum_{i=0}^\infty [\pi^\flat]^i z_i,$$

and since $W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)$ is $[\pi^\flat]$-adically complete this finishes the proof (see for example [Bhatt, Morrow, Scholze: Integral p-adic Hodge theory, Lemma 3.2(ii)]). $\square$

Lemma 11. *Let C be as above. Then $\mathcal{O}_C^\flat$ is a valuation ring with associated valuation*

$$\nu^\flat\colon \mathcal{O}_C^\flat \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto \nu(x^\sharp).$$

Moreover, $\mathcal{O}_C^\flat$ is complete for its valuation topology and its fraction field $C^\flat := \mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed.

Proof. We first show that $\nu^\flat$ is a valuation:

1. Since $(-)^\sharp$ is multiplicative, we have $\nu^\flat(xy) = \nu^\flat(x) + \nu^\flat(y)$.
2. $\nu^\flat(x) = \infty$, iff $x^\sharp = 0$ in $\mathcal{O}_C$, iff $x_i = 0 \in \mathcal{O}_C/(\pi)$ for all i, where $x = (x_0, x_1, \dots)$.

3. For the non-archimedian triangle inequality, we use that we know from earlier that $(x^{1/q^n})^\sharp$ is a lift of x_n modulo π. Therefore, we have

$$\begin{aligned}
\nu^\flat(x+y) &= \nu((x+y)^\sharp) = \nu(\lim_{n\to\infty}((x^{1/q^n})^\sharp + (y^{1/q^n})^\sharp)^{q^n}) \\
&= \lim_{n\to\infty} q^n \nu((x^{1/q^n})^\sharp + (y^{1/q^n})^\sharp) \\
&\geq \lim_{i\to\infty} \min(\nu((x^{1/q^n})^\sharp), \nu((y^{1/q^n})^\sharp)) \\
&= \lim_{n\to\infty} \min(\nu(((x^{1/q^n})^\sharp)^{q^n}), \nu(((y^{1/q^n})^\sharp)^{q^n})) \\
&= \min(\nu^\flat(x), \nu^\flat(y))
\end{aligned}$$

The next thing, we want to show is the completeness. First note that the inverse limit topology induced by $\mathcal{O}_C^\flat \cong \lim_{x\mapsto x^q} \mathcal{O}_C$ is complete, since C is completely valued ($\mathcal{O}_C$ is a Fréchet space and $\lim_{x\mapsto x^q} \mathcal{O}_C \subset \prod_{n\in\mathbb{N}} \mathcal{O}_C$ is closed. Then $\prod_{n\in\mathbb{N}} \mathcal{O}_C$, as a countable product of Fréchet spaces, is again a Fréchet space and the closed subset $\lim_{x\mapsto x^q} \mathcal{O}_C$ is complete.)

But a basis for the inverse limit topology around 0 is given by sets of the form

$$\{x \in \mathcal{O}_C^\flat \,|\, \nu((x^{1/q^n})^\sharp) \geq m\} = \{x \in \mathcal{O}_C^\flat \,|\, \nu^\flat(x) \geq q^n m\}$$

for $n, m \geq 0$ and a basis for the topology induced by $\nu^\flat$ is given by the sets

$$\{x \in \mathcal{O}_C^\flat \,|\, \nu^\flat(x) \geq m\}$$

for $m \geq 0$, which shows that the topologies are equivalent.

It remains to show that $C^\flat$ is algebraically closed. To show this, it suffices to show that every normed polynomial in $\mathcal{O}_C^\flat[T]$ of degree ≥ 1 has a root. (Because: Let $f = q_d T^d + \cdots + q_0 \in C^\flat[T]$ with $q_n \neq 0$. By multiplying with nonzero elements in $\mathcal{O}_C^\flat$, we can assume that $q_i \in \mathcal{O}_C^\flat$. But now f has a zero, if and only if $q_d^{d-1} f = (q_d T)^d + q_{d-1}(q_d T)^{d-1} + q_{d-2} q_{d-1} (q_d T)^{d-2} + \cdots + a_0 q_n^{d-1}$ has a zero, and this has a zero for T if and only if it has a zero for $q_d T$.)
So let

$$f = T^d + a_{d-1} T^{d-1} + \cdots + a_0 \in \mathcal{O}_C^\flat[T]$$

with $d \geq 1$ and we want to show that it has a zero in $\mathcal{O}_C^\flat$. We set

$$f_n(T) \coloneqq T^d + (a_{d-1}^{1/q^n})^\sharp T^{d-1} + \cdots + (a_0^{1/q^n})^\sharp \in \mathcal{O}_C[T].$$

Then

$$\begin{aligned}
f_{n+1}(T)^q &= (T^d)^q + ((a_{d-1}^{1/q^{n+1}})^\sharp T^{d-1})^q + \cdots + ((a_0^{1/q^{n+1}})^\sharp)^q + q \cdot (\dots) \\
&= T^d q + (a_{d+1}^{1/q^n})^\sharp T^{(d-1)q} + \cdots + (a_0^{1/q^n})^\sharp + q \cdot (\dots) \\
&\equiv f_n(T^q) \mod \pi.
\end{aligned}$$

Let us now fix an $n \geq 0$, let $x \in \mathcal{O}_C$ be a zero of f_n and let $y \in \mathcal{O}_C$ such that $y^q = x$. Since C is algebraically closed, we know that x and y exist. By the above observation, we have

$$f_{n+1}(y)^q \equiv f_n(y^q) = f_n(x) = 0 \mod \pi,$$

which means that π divides $f_{n+1}(y)^q$, and since we are in a valuation ring, this means that

$$\nu(f_{n+1}(y)) \geq \frac{1}{q} v(\pi).$$

Let $z_1, \ldots, z_d \in \mathcal{O}_C$ be the zeros of f_{n+1}. Again we know that they exist. Then

$$f_{n+1} = \prod_{i=1}^{d} (T - z_i),$$

and therefore

$$f_{n+1}(y) = \prod_{i=1}^{d} (y - z_i).$$

Since

$$\nu(f_{n+1}(y)) = \sum_{i=1}^{d} \nu(y - z_i) \geq \frac{1}{q} \nu(\pi),$$

there must be an i such that $\nu(y - z_i) \geq \frac{1}{dq} \nu(\pi)$. This means that π divides

$$(y - z_i)^{qd} = ((y - z_i)^q)^d = (y^q - z_i^q + qa)^d = (y^q - z_i^q)^d + qa',$$

for some $a, a' \in \mathcal{O}_C$. But since π divides q, we see that π has to divide $(y^q - z_i^q)^d$, which implies

$$\nu(x - z_i^q) \geq \frac{1}{d} \nu(\pi).$$

Inductively, we get a sequence $(x_n)_{n \geq 0} \subset \mathcal{O}_C$ such that

- $f_n(x_n) = 0$
- $\nu(x_{n+1}^q - x_n) \geq \frac{1}{d} \nu(\pi)$.

If we now set $\mathfrak{a} := \{y \in \mathcal{O}_C \mid \nu(y) \geq \frac{1}{d} \nu(\pi)\}$, which is an ideal in $\mathcal{O}_C$, then $x := (x_0, x_1, \ldots)$ defines an element in

$$(\mathcal{O}_C/\mathfrak{a})^\flat \cong \varprojlim_{x \mapsto x^q} \mathcal{O}_C \cong \mathcal{O}_C^\flat,$$

and clearly $f(x) = 0$. $\square$