FARGUES–FONTAINE CURVE: NEWTON POLYGONS II

VINCENT GAJDA

Abstract. These are lecture notes for a talk given in a seminar on the Fargues–Fontaine curve, which took place in Wuppertal, WS 2025/26. We discuss some general facts about Newton polygons, construct a metric on the space of untilts $|Y|$ and then show how to use Newton polygons to find linear factors of elements in $\mathbb{A}_{\mathrm{inf}}$. We closely follow the lecture notes of Anschütz [1], which are based on [2], in particular nothing here is original work.

1. Newton Polygons for $\mathbb{A}_{\mathrm{inf}}$

The classical Newton polygon of a polynomial can be seen as a simple visualization of (valuation) data associated to the coefficients of the polynomial. Its definition can be extended to also encompass power series and we shall start this talk by extending its definition even further to elements of Fontaine's first period ring $\mathbb{A}_{\mathrm{inf}}$. Recall that to define this ring we fixed some prime p, a finite extension $E/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and F a non-archimedean, complete, algebraically closed extension of $\mathbb{F}_q$. Furthermore let π be a fixed uniformizer of the ring of integers $\mathcal{O}_E$. With these choices we defined $\mathbb{A}_{\mathrm{inf}} := W_{\mathcal{O}_E}(\mathcal{O}_F)$ as the Witt algebra of $\mathcal{O}_F$ over $\mathcal{O}_E$. This ring was discussed plenty in the previous talks and in particular we saw that every element f of $\mathbb{A}_{\mathrm{inf}}$ has a unique Teichmüller expansion

$$f = \sum_{n\geq 0} [a_n]\pi^n,$$

with $a_n \in \mathcal{O}_F$, π the uniformizer and the brackets denoting the Teichmüller lift. For the definition of the Newton polygon of such an f we need the following valuations.

Definition 1.1. For each $r \geq 0$ and $f = \sum_{n\geq 0}[a_n]\pi^n \in \mathbb{A}_{\mathrm{inf}}$ we define

$$\nu_r(f) := \inf_{n\in\mathbb{N}} \{\nu(a_n) + rn\},$$

where ν denotes the valuation on $\mathcal{O}_F$.

Definition 1.2. We define $\mathrm{Newt}(f)$ of an element $f = \sum_{n\geq 0}[a_n]\pi^n \in \mathbb{A}_{\mathrm{inf}}$ to be the maximal, convex, decreasing polygon in $\mathbb{R}^2$ which lies under the points $(n, \nu(a_n))_{n\geq 0}$.

Remark 1.3. If we blur the line between polygons and piecewise-linear functions we can also give the following definition: recall that $\mathcal{F}$ was the set of functions from $\mathbb{R}$ to $\hat{\mathbb{R}} := \mathbb{R} \cup \infty \cup -\infty$. We define $\mathrm{Newt}(f)$ to be the maximal, decreasing, convex function in $\mathcal{F}$, whose Legendre transform is given by

$$\mathcal{L}(\mathrm{Newt}(f))(r) := \begin{cases} \nu_r(f), & r \geq 0 \\ -\infty, & r < 0 \end{cases}$$

We recall that the Legendre transform of a function $\phi \in \mathcal{F}$ is defined as

$$\mathcal{L}(\phi)(r) := \inf_{x \in \mathbb{R}} \{\phi(x) + rx\}.$$

Just as for the classical Newton-polygon we obtain a certain multiplicativity result.

Proposition 1.4. *The maps ν_r are valuations. In particular for $f, g \in \mathbb{A}_{\mathrm{inf}}$ we have*

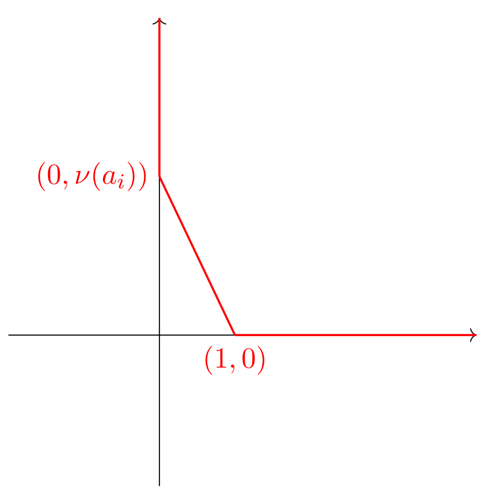
$$\mathrm{Newt}(f \cdot g) = \mathrm{Newt}(f) * \mathrm{Newt}(g),$$

where $*$ denotes the convolution product on elements of $\mathcal{F}$ which never take the value $-\infty$. It is defined by:

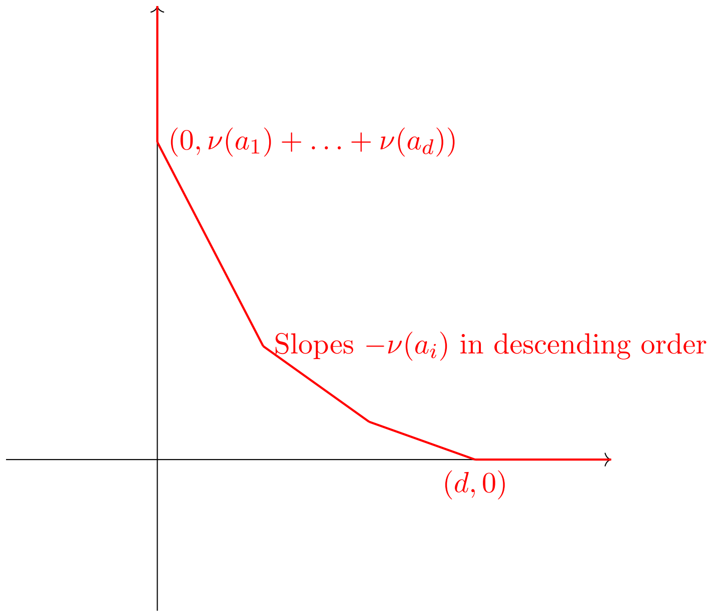
$$(\phi * \psi)(x) := \inf_{a+b=x} \{\phi(a) + \psi(b)\}.$$

Proof. See [1, Proposition 6.22] for a sketch or [2, Section 1.4.2]. □

Example 1.5. Let $a_1, \ldots, a_d \in \mathfrak{m}_F \backslash \{0\}$. Set $f = (\pi - [a_1]) \cdot \ldots \cdot (\pi - [a_d]) \in \mathbb{A}_{\mathrm{inf}}$. Then by definition $\mathrm{Newt}(\pi - [a_i])$ consists of an infinitely steep slope, one section of slope $-\nu(a_i)$ and one flat slope:



and via the previous Proposition $\mathrm{Newt}(f)$ is a polygon which contains all these slopes in a descending order:



2. The metric space $|Y|$

The main goal of this talk is to prove the following

Main Theorem. *Let $f = \sum_{n\geq 0}[a_n]\pi^n \in \mathbb{A}_{\mathrm{inf}}$ and let $\lambda \neq 0$ be a (finite) slope of* $\mathrm{Newt}(f)$*. Then there exists $a \in \mathfrak{m}_F$ with $\nu(a) = -\lambda$ and $g \in \mathbb{A}_{\mathrm{inf}}$ such that $f = (\pi - [a]) \cdot g$.*

This can be seen as an $\mathbb{A}_{\mathrm{inf}}$-variant of an analogous classical theorem of Lazard for a power series ring. The classical theorem essentially tells us the following. Consider the power series expansion of an analytic function around 0. If we find a slope $\lambda \neq 0$ in the Newton polygon of this power series expansion, we can already deduce the existence of zeroes with valuation $-\lambda$. We want to have an analogous interpretation of the Main Theorem. For this we want to interpret elements of $\mathbb{A}_{\mathrm{inf}}$ as functions on a certain adic variant of the (punctured) open unit disc, namely on the space of characteristic 0 untilts $|Y|$ from a previous talk. Furthermore we want to endow $|Y|$ with a metric, in order to consider Cauchy-sequences of points converging to zeroes. Let us recall some definitions.

Definition 2.1. We set

$$\begin{aligned} |Y|_{[0,\infty)} &:= \{\text{Untilts of } \mathcal{O}_F\}/_{\cong} \\ &\cong \{I \subseteq \mathbb{A}_{\mathrm{inf}} \mid (\mathbb{A}_{\mathrm{inf}}, I) \text{ is a perfect prism}\}/_{\cong} \end{aligned}$$

$$\begin{aligned} |Y| &:= |Y|_{[0,\infty)} \backslash \{(\pi)\} \\ &\cong \{\text{Characteristic 0 untilts of } \mathcal{O}_F\}/_{\cong} \\ &\cong \mathrm{Prim}_1/\mathbb{A}_{\mathrm{inf}}^{\times} \end{aligned}$$

Here for $d \geq 0$ Prim_d denotes the set of primitive elements of degree d, i.e.

$$\mathrm{Prim}_d := \{f = \sum_{n\geq 0}[a_n]\pi^n \mid a_0 \neq 0,\ \text{d minimial s.t. } a_d \in \mathcal{O}_F^\times\}$$

Remark 2.2. Note that by the remarks following [1, Definition 4.4] for every element in $|Y|_{[0,\infty)}$ we can choose the generator of the corresponding I to be of the form $(\pi - [a])$ for some $a \in \mathfrak{m}_F$. The unique characteristic p untilt corresponds to $a = 0$.

We now want to endow the set $|Y|$ with some geometry. As a side remark the following definitions are a remnant of the fact that $|Y|$ can also be defined as the set of classical points of the adic space

$$\mathcal{Y} := \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}})\backslash\{\pi[\varpi]\}.$$

Definition 2.3. Let $y \in |Y|$. Then we define:

(1) $\mathfrak{p}_y$ to be the prime ideal associated to y as part of the perfect prism corresponding to y,
(2) ξ_y to be a generator of $\mathfrak{p}_y$, which we can always choose to be of the form $\xi_y = \pi - [a]$ for some $a \in \mathfrak{m}_F\backslash\{0\}$,
(3) $C_y := (\mathbb{A}_{\mathrm{inf}}/\mathfrak{p}_y)\left[\frac{1}{\pi}\right]$ the *residue field at* y,
(4) $\theta_y : \mathbb{A}_{\mathrm{inf}} \to C_y$ the natural map,
(5) $\nu_y : C_y \to \mathbb{R} \cup \infty$ the valuation defined by

$$\nu_y(\theta_y([x])) = \nu(x),$$

where ν is the valuation on $\mathcal{O}_F$. This is well-defined, since every element in $\mathbb{A}_{\mathrm{inf}}/\mathfrak{p}_y$ can be written as the projection of a Teichmüller lift of some $x \in \mathcal{O}_F$, which was mentioned in Daan's talk, see also [1, Corollary 5.3].
(6) For $f \in \mathbb{A}_{\mathrm{inf}}$ we set

$$f(y) := \theta_y(f),\ \nu(f(y)) := \nu_y(\theta_y(f)),$$

so that we can now make sense of f as a function on $|Y|$ which takes value $f(y) = \theta_y(f)$ at y.

In addition to these algebro-geometric notions we want to define a metric on $|Y|$.

Definition 2.4. For $y_1, y_2 \in |Y|$ set

$$d(y_1, y_2) := \nu_{y_1}(\theta_{y_1}(\xi_{y_2})),\ d(y_1, 0) := \nu(\pi(\xi_{y_1})),$$

where $\pi : \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_F$ is the natural projection.

To prove that this defines a metric one needs for every $r \geq 0$ the following ideals in $\mathbb{A}_{\mathrm{inf}}$:

$$\mathfrak{a}_r := \{f \in \mathbb{A}_{\mathrm{inf}} \mid \nu_0(f) = \inf\{\nu(a_i)\} \geq r\}.$$

Proposition 2.5. *For $y_1, y_2, y_3 \in |Y|$ we have that*

$$d(y_1, y_2) = \sup\{r \mid \mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r\}.$$

In particular $d(-,-)$ defines a "valuative" ultra-metric, i.e.

(1) $d(y_1, y_2) = d(y_2, y_1)$,
(2) $d(y_1, y_3) \geq \min\{d(y_1, y_2), d(y_2, y_3)\}$,
(3) $d(y_1, y_2) = \infty \iff y_1 = y_2$.

Proof. See [1, Lemma 7.4] or [2, Proposition 2.3.2] for sketches. We remark here that the metric with above definition of $d(y, 0)$ extends to $|Y|_{[0,\infty)}$. □

Proposition 2.6. *Set for $r > 0$*

$$|Y_r| := \{y \in |Y| \mid d(y, 0) = r\}.$$

Then $|Y_r|$ is a complete metric space with the restricted metric from the previous proposition. Moreover, if we set for any closed interval $I \subseteq (0, \infty)$

$$|Y_I| := \bigcup_{r \in I} |Y_r|,$$

then also any such $|Y_I|$ is complete

Proof. See [1, Proposition 7.5] or [2, Proposition 2.3.4]. Note that Fargues–Fontaine use $\leq r$ instead of $= r$ in the definition of $|Y_r|$, but this does not matter for the discussion. □

3. Proof of the Main Theorem

We can now give the proof of the main theorem. The proof given here is simply a reformulation of the proof in [2, Section 2.4, in particular Theorems 2.4.1 and 2.4.5], see also [1, Theorem 7.1]. Note that one should really add *finite* slope, see the discussion in the proof.

Main Theorem. *Let $f = \sum_{n \geq 0} [a_n] \pi^n \in \mathbb{A}_{\mathrm{inf}}$ and let $\lambda \neq 0$ be a (finite) slope of* $\mathrm{Newt}(f)$. *Then there exists $a \in \mathfrak{m}_F$ with $\nu(a) = -\lambda$ and $g \in \mathbb{A}_{\mathrm{inf}}$ such that $f = (\pi - [a]) \cdot g$.*

Proof. Step 1: we first will reduce to the case that f is primitive of some degree $d \geq 0$. For this we write

$$f = \sum_{n \geq 0} [a_n] \pi^n = \lim_{d \to \infty} f_d$$

with $f_d = \sum_{n=0}^{d} [a_n] \pi^n$ being the truncation of f at degree d. Now f_d is not necessarily a primitive element, but it is pretty close to being one. More precisely by [3, Lecture 15, Remark 5] we can write

$$f_d = \pi^k \cdot [c] \cdot f'_d$$

where f'_d is a primitive element of some degree $d' \leq d$. As a side remark, k and $\nu(c)$ are uniquely determined, but the decomposition itself is not.

Now consider the set

$$X_d := \{y \in |Y| \mid f_d(y) = 0,\ d(y, 0) = \nu(\pi(\xi_y)) = \nu([a_y]) = -\lambda\},$$

where we used that we could always choose the generator ξ_y of $\mathfrak{p}_y$ to be of the form $\pi - [a_y]$ for some $a_y \in \mathfrak{m}_F$. In other words X_d is the set of zeroes of f_d (i.e. the set of zeroes of f'_d as above) with valuation $-\lambda$. Note in particular that this set is a subspace of the complete space $|Y_{-\lambda}|$.

Assume that each X_d is not empty. Then as discussed in [2, Theorem 2.4.5] one can find a Cauchy-sequence $\{y_d\}_{d\geq 0}$ with $y_d \in X_d$. By Proposition 2.5 this sequence converges to some $y \in |Y|$. Now by continuity properties of the involved valuations discussed in [2, Section 2.3.3] this limit-point is a zero of f. More precisely one needs here and later in this proof that $\nu_y(f)$ is continuous in $y \in |Y|$ and $f \in \mathbb{A}_{\text{inf}}$, which follows from the discussion and proofs in the reference.

Step 2: now we will prove that X_d is not empty. In other words, we simply have to prove the main theorem for the special case that f is primitive of some degree $d \geq 0$. Note that $d = 0$ implies that every non zero slope of $\mathrm{Newt}(f)$ is $-\infty$ and these slopes are not considered in [1] (probably because they are not drawn if you consider the polygon as a function). In particular, I explained this case wrong in the talk. Indeed, the statement would be wrong if we include infinite slopes, as for example $[a] \in \mathbb{A}_{\text{inf}}$ for $a \in \mathcal{O}_F^\times$ cannot be factored as $[a] = \pi \cdot g$ for any $g \in \mathbb{A}_{\text{inf}}$, by uniqueness of the Teichmüller expansion. There is no issue in the way Fargues–Fontaine prove the statement as they are already working in a localization of $\mathbb{A}_{\text{inf}}$, where π is invertible. As this is the setting for which we will use the theorem as well, there shall be no problems. Either way, we can assume $d \geq 1$.

We can without loss of generality assume that λ is the steepest slope of $\mathrm{Newt}(f)$. Indeed, since f is primitive of degree it only has at most d non-trivial slopes. By induction we can factor work our way through steeper (finite) slopes until we reach the desired slope λ.

Now we want to construct a sequence of $y_n \in |Y|$ such that:

(1) $\nu(f(y_n)) \geq -(d+n)\lambda$,
(2) $d(y_n, y_{n+1}) \geq -\frac{d+n}{d}\lambda$,
(3) $\nu(\pi(y_n)) = d(y_n, 0) = -\lambda$.

In particular, this sequence is a Cauchy-sequence by the second property. It thus converges to a y with $\nu(\pi(y)) = -\lambda$ and by $\nu(f(y_n)) \geq -(d+n)\lambda$ we have $f(y) = 0$, i.e. we have found the y demanded by the theorem.

Now write $f = \sum_{n\geq 0}[x_n]\pi^n$. As f is primitive of degree d we have $x_d \in \mathcal{O}_F^\times$. Since F is algebraically closed and because $\mathcal{O}_F \subseteq F$ is integrally closed we can find a zero $z \in \mathcal{O}_F$ of

$$f' = \sum_{n\geq 0}^{d} x_n T^n,$$

which we assume to be of maximal valuation among all zeroes of f'. By Dzoara's talk we know that the valuation of the zeroes of f' corresponds to the slopes of the Newton polygon of f'. Because the Newton polygon of f reaches zero at d these slopes are exactly the slopes of $\mathrm{Newt}(f)$. In particular the valuation of z is $-\lambda$. Set $y_1 = (\pi - [z]) \in |Y|$. As z had the maximal valuation $-\lambda$ among all zeroes of f' we see using basic properties of valuations that

$$\nu(x_i) \geq -\lambda(d-i) \text{ for all } 0 \leq i \leq d.$$

This implies that the unique $w_i \in F$ with

$$x_i z^i = w_i z^d$$

already lies in $\mathcal{O}_F$. We deduce

$$f(y_1) = \nu_{y_1}(\theta_{y_1}(f)) = \nu_{y_1}\left(\theta_{y_1}\left(\sum_{n=0}^{d}[x_n]\pi^n + \pi^{d+1}\sum_{n\geq d+1}[x_n]\pi^{n-d-1}\right)\right)$$

By the above discussion and using that $\pi = \theta_{y_1}(\pi) = \theta_{y_1}([z])$ we have

$$\begin{aligned}\theta_{y_1}(\sum_{n=0}^{d}[x_n]\pi^n) &= \sum_{n=0}^{d}\theta_{y_1}([x_n\cdot z^n]) = \sum_{n=0}^{d}\theta_{y_1}([w_n\cdot z^d]) \\ &= \pi^d\cdot\sum_{n=0}^{d}\theta_{y_1}([w_n])\end{aligned}$$

We have $\sum_{n=0}^{d} w_i = 0$, because $\sum_{n=0}^{d} w_i\cdot z^d = \sum_{n=0}^{d} x_i z^i = 0$ and $\mathcal{O}_F$ has no zero-divisors. Hence, $\sum_{n=0}^{d}[w_i]$ lies in the kernel of the projection $\pi : \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_F$, i.e. $\sum_{n=0}^{d}[w_i] \in \pi\mathbb{A}_{\mathrm{inf}}$. In total we see now that $\theta_{y_1}(f) \in \pi^{d+1}\mathbb{A}_{\mathrm{inf}}/\mathfrak{p}_{y_1}$ and therefore $f(y_1) = \nu_{y_1}(\theta_{y_1}(f)) \geq -(d+1)\lambda$. This finishes the induction start, i.e. the first element of our sequence y_1 has been constructed and proven to have the correct properties.

Let us discuss some ideas for the induction step. So, assume y_n as desired has been constructed. Essentially the induction step is very analogous to the induction start, one simply has to redo the arguments with respect to the "local coordinate" $\xi = \pi - [b_n]$ corresponding to y_n instead of the "local coordinate" π, which we used in the induction start. So let

$$f = \sum_{n\geq 0}[a_n]\xi^n$$

be the (non-unique) ξ-adic expansion of f, which exists by [2, Corollaire 2.2.12]. By using the projection down to $W_{\mathcal{O}_E}(\mathcal{O}_F/\mathfrak{m}_F)$ and by the analogous properties for the π-adic expansion of f we can deduce that $a_d \in \mathcal{O}_F^\times$ and $\nu(a_i) > 0$ for $0 \leq i < d$. Now the inductive fact that $\nu(f(y_n)) = \nu_{y_n}(\theta_{y_n}(f)) \geq -(d+n)\lambda$ translates to

$$\nu(a_0) \geq -(d+n)\lambda.$$

Let $z \in \mathcal{O}_F$ be a zero of $f' = \sum_{n=0}^{d} a_n T^n$ of maximal valuation. Then using the explicit definition of the addition in $\mathbb{A}_{\mathrm{inf}}$ one can deduce that $\xi - [z]$ is still primitive of degree 1 so that we can set

$$y_{n+1} = (\xi - [z]) \in |Y|.$$

z being of maximal valuation among all zeroes of f' now gives

$$d(y_n, y_{n+1}) = \nu(z) \geq \frac{\nu(a_0)}{d} \geq -\frac{d+n}{d}\lambda.$$

By the ultra-metric triangle inequality we get in particular $d(y_{n+1}, 0) = \nu_{y_{n+1}}(\pi) = -\lambda$.

The only thing left to show is $\nu(f(y_{n+1})) \geq -(d+n+1)\lambda$. Since z had maximal valuation among all zeroes of f' we deduce $\nu(a_i) + i\nu(z) \geq \nu(a_0)$ for $0 \leq i \leq d$. Similar to before we find $b_i \in \mathcal{O}_F$ with $b_i a_0 = a_i z^i$ and again we calculate

$$f(y_{n+1}) = \theta_{y_{n+1}}([a_0]) \sum_{n=0}^{d} \theta_{y_{n+1}}([b_n]) + \theta_{y_{n+1}}([z]^{d+1}) \sum_{n \geq d+1} \theta_{y_{n+1}}([a_n z^{n-d-1}]).$$

We deduce as before

$$\begin{aligned} \nu_{y_{n+1}}(\theta_{y_{n+1}}([a_0]) \cdot \theta_{y_{n+1}}(\sum_{n=0}^{d}[b_i])) &\geq \nu(a_0) + \nu_{y_{n+1}}(\pi) \\ &\geq -(d+n)\lambda - \lambda = -(d+n+1)\lambda \end{aligned}$$

and also

$$\begin{aligned} \nu_{y_{n+1}}(\theta_{y_{n+1}}([z]^{d+1})) &= (d+1)\nu(z) \\ &\geq -\frac{(d+1)(d+n)}{d}\lambda \geq -(d+n+1)\lambda, \end{aligned}$$

from which we can deduce the last remaining claim. □

As a short outlook: the main theorem here turns out to be a technical lemma, which allows us to factorize certain elements in a graded ring related to $\mathbb{A}_{\mathrm{inf}}$. Via the Proj-construction this graded ring will give rise to the (schematic) Fargues–Fontaine curve and the factorization will be useful in proving that the curve is a Dedekind-scheme.

References

[1] Johannes Anschütz. *Lectures on the Fargues-Fontaine Curve.*

[2] Laurent Fargues and Jean-Marc Fontaine. *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* Asterisque, 2018.

[3] Jacob Lurie. *The Fargues-Fontaine Curve.* 2018. URL: `https://www.math.ias.edu/œlurie/`.