The multiplicative structure of P

Recall: For $d \in \mathbb{Z}$, considered

$$P^d := B^{\varphi = \pi^d} := \{ b \in B : \varphi(b) = \pi^d b \}$$

and $$P := \bigoplus_{d \in \mathbb{Z}} P^d \rightsquigarrow X := \mathrm{Proj}(P)$$

is called schematic

FF - curve

Plan:

* Prove some properties of B_+

* Use them to show that the graded alg P is factorial and enjoys "nice properties"

§1: The rings B_I: $I \subseteq (0, \infty)$ interval

Prop 1 The ring B_I is an integral domain

Pf:

This follows from a result of last time:

Key result: If (f_n) seq in B^b s.t. $f := \lim_n f_n \neq 0$ in B_I which is Cauchy wrt every v_r $(r \in I)$ then $\exists N \gg 0$ s.t. $\forall n \geq N$ and all $r \in I$

$$v_r(f_n) = v_r(f_N) < \infty.$$

$\square$

Thm 2: If I is cpct, then B_I is PID.

To prove Thm 2, we need some results.

Lemma 3: Let $y \in |Y_I|$

① The projection $\theta_y : B^b \to C_y := \left(A_{\inf} / \gamma_y\right)[\frac{1}{\pi}]$

extends to $\theta_y^I : B_I \to C_y$.

② The kernel of θ_y^I is generated by ξ_y.

Pf: Set $r := d(y, 0)$.

① By univ property, STS that θ_y is cont for the top induced by v_r.

Note for $x = \sum_{i \gg -\infty} [x_i]\pi^i \in B^b$

$$\rightsquigarrow \quad \theta_y(x) = \sum_{i \gg -\infty} \theta_y([x_i])\pi^i \quad \text{and hence}$$

$$v_y(\theta_y(x)) \geq \inf_{i\in\mathbb{Z}} \{ v_y(\theta_y([x_i])\pi^i) \}$$

$$= \inf_{i\in\mathbb{Z}} \{ v(x_i) + i \underset{\underset{1}{\shortparallel}}{v_y(\pi)} \}$$

$$= v_1(x)$$

This means that if $(f_n)_n \subseteq B^b$ is Cauchy wrt v_1, then $(\theta_y(f_n))_n \in C_y$ is Cauchy wrt v_y $\Rightarrow$ θ_y is continuous.

② As $\ker \theta_y = \mathcal{E}_y B^b$

$\Rightarrow \ker \theta_y^I = \overline{\mathcal{E}_y B^b}$

one direction is clear, the other uses π-adic top on C_y

We show: $\overline{\mathcal{E}_y B^b} = \mathcal{E}_y B_I$

"$\subseteq$" Let $f \in \overline{\mathcal{E}_y B^b}$, then $f = \lim_n f_n$

where $f_n = \mathcal{E}_y g_n$ with $g_n \in B^b$ $\forall n$.

Note $v_2(f_m - f_n) = v_2(g_m - g_n) + \underset{\underset{\infty}{\neq}}{v_2(\mathcal{E}_y)}$

So (f_n) Cauchy $\Rightarrow$ (g_n) Cauchy

"$\supseteq$" As $\mathcal{E}_y \in \ker \theta_y^I \Rightarrow \mathcal{E}_y B_I \subseteq \ker \theta_y^I$ $\square$

Lemma 4: We have

$$B_I^\times = \{ f \in B_I \setminus \{0\} : \mathrm{Newt}_I(f) = \emptyset \}.$$

Pf: A_1

- $B_I = \varprojlim_{\substack{J \subseteq I \\ \text{cpct}}} B_J$

- $\mathrm{Newt}_I(f) = \bigcup_{\substack{J \subseteq I \\ \text{cpct}}} \mathrm{Newt}_J(f)$

STS statement for $I = [a, b]$ cpct.

In this case, "$\subseteq$" is clear.

$$\left(\mathrm{Newt}_I^{\,0}(u) * \mathrm{Newt}_I^{\,0}(u^{-1}) = \mathrm{Newt}_I^{\,0}(1) = \emptyset \right)$$

"$\supseteq$" Write $f = \lim_n f_n$ for $f_n \in B^b$, then

$\exists N \gg 0$ s.t. $\mathrm{Newt}_I(f_n) = \mathrm{Newt}_I(f) \;\; \forall n \geq N.$

(see Key Result) $\qquad\qquad = \emptyset$

Hence, we may also reduce to $f \in B^b$.

Write

$$f = \sum_{n \gg -\infty} [x_n] \pi^n = \underbrace{\sum_{n > N} [x_n] \pi^n}_{=: z} + \underbrace{\sum_{n \leq N} [x_n] \pi^n}_{=: y}$$

with N s.t.

* each slope of Newt(f) on $(-\infty, N)$ is strictly less than $-b$

* " ―――――――――――――― " (N, ∞) is strictly greater than $-a$.

(this is possible as $\mathrm{Newt}_{[a,b]}(f) = \emptyset$)

Let λ_1 be slope of Newt(f) on $[N-1, N]$

λ_2 " ―――――――――― " $[N, N+1]$

For $n \leq N$:

$$\frac{v(x_N) - v(x_n)}{N-n} \leq \lambda_1$$

$$\Leftrightarrow \quad v(x_n) \geq (n-N)\lambda_1 + v(x_N) \quad (*)$$

Similarly for $n \geq N$,

$$v(X_n) \geq (n-N)\lambda_2 + v(X_N) \quad (**)$$

N is a break-pt of $\mathrm{Newt}(f)$, hence $X_N \neq 0$,

and we can write

$$y = [X_N]\pi^N \Big(\underbrace{1 + \sum_{-\infty << n < N} [X_n X_N^{-1}]\pi^{n-N}}_{\tilde{y}}\Big)$$

<u>Claim:</u> $\tilde{y}$ is top nilpotent in B_I

(its powers converge to 0 wrt all $|\cdot|_r$ for $r \in I$)

<u>Pf Claim:</u> For $r \in I = [a, b]$

$$v_\tau(\tilde{y}) = \inf_{n<N} \{ v(x_n x_N^{-1}) + (n-N)\tau \}$$

$$⊛ \rightarrow \;\geqslant \inf_{n<N} \{ (n-N)\lambda_1 + \underbrace{v(x_N) - v(x_N) + (n-N)\tau}_{} \}$$

$$(n-N)(\lambda_1+\tau)$$

$$= -\lambda_1 - \tau > 0$$

$$\uparrow$$

as

$$-\lambda_1 > b \geqslant \tau \Rightarrow 0 > \lambda_1 + \tau$$

Hence, $y = [x_N]\pi^N(1+\tilde{y}) \in B_I^\times$ and

$$f = y(1+y^{-1}z)$$

Here, we are using $[x_N]\pi^N \in B_I^\times$.

Claim: $y^{-1}Z$ is top nilp in B_I

Pf Claim: Similar, use (**) ☒

So, $f \in B_I^\times$ □

Thm 5: (Lazard's Thm for B_I)

Let $f \in B_I$. If λ is a slope of $\mathrm{Newt}_I(f)$ then there exists a $a \in m_F$ with $v(a) = -\lambda$ and $f = (\pi - [a])g$ for some $g \in B_I$.

Pf Sketch:

By Lemma 3, STS $\exists y \in |Y_{-\lambda}|$ st

$$f \in \ker(\Theta_y^I) = \mathcal{E}_y B_I .$$

Construct y via an approximation argument, like in Vincent's talk ($|Y_{-\lambda}|$ is complete).

$\square$

Proof of Theorem 2:

Suppose I is cpct

We will use:

Lemma: An int dom A is a PID $\Leftrightarrow$

① A is a UFD

and

② Every irred nonunit $x \in A$ generates a max ideal.

Use A PID $\Leftarrow$ every prime ideal principal to see this

① B_I UFD:

Let $f \in B_I \setminus B_I^\times$. As I cpct, $\mathrm{Newt}_I(f)$ has fin many slopes.

By Lemma 4, $\mathrm{Newt}_I(f) \neq \emptyset$.

Let λ be a slope of $\mathrm{Newt}_I(f)$.

By Thm 5, $f = \varepsilon_y g$ for some $g \in B_I \setminus \{0\}$ and $y \in |Y_{-\lambda}|$.

$\Rightarrow \mathrm{Newt}_I(g)$ has the same slopes as $\mathrm{Newt}_I f$, with the multiplicity of λ one less.

Iterating:

$$f = u\,\mathcal{E}_{y_1} \cdots \mathcal{E}_{y_n} \quad \text{for some } y_i \in |Y_I|$$

and $u \in B_I$ st $\mathrm{Newt}_I(u) = \emptyset \Rightarrow u \in B_I^\times$
by Lemma 4.

$\Rightarrow B_I$ is UFD.

(2) STS $\mathcal{E}_y$ generates a max ideal

By Lemma 3: $B_I / \mathcal{E}_y B_I \cong C_y$ ✓ $\square$

§2: Divisors and the structure of P

Recall: For $y \in |Y_I|$, $B^+_{dR,y} := (\mathbb{A}_{inf}[\frac{1}{\pi}])^\wedge_{\mathcal{E}_y}$

is complete DVR with residue field C_y.

One can show: $B^+_{dR,y} \cong (B^b)^\wedge_{\mathcal{E}_y} \cong (B_I)^\wedge_{\mathcal{E}_y}$

using Lemma 3.

Def 6:

① For I cpct, consider the (abelian) monoid

$$\mathrm{Div}^+(|Y_I|) := \mathbb{N}[|Y_I|].$$

(2) For general I, define

$$\mathrm{Div}^+(|Y_I|) := \lim_{\substack{\longleftarrow \\ J \subseteq I \text{ cpct}}} \mathrm{Div}^+(|Y_J|).$$

So, $\mathrm{Div}^+(|Y_I|)$ is the monoid of formal sums $\sum_{y \in |Y_I|} n_y y$, $n_y \in \mathbb{N}$ s.t.

for each cpct $J \subseteq I$, $|\{y \in |Y_J| : n_y \neq 0\}| < \infty$.

We have a partial order on $\mathrm{Div}^+(|Y_I|)$:

$$\sum_{y \in |Y_I|} n_y y \geq \sum_{y \in |Y_I|} m_y y \text{ iff } n_y \geq m_y \ \forall y \in |Y_I|.$$

(3) Recall that φ acts on $|Y|$ by $y \mapsto \varphi(y)$. Define

$$\mathrm{Div}^+\left({|Y|}/{\varphi^{\mathbb{Z}}} \right) := \{ D \in \mathrm{Div}^+(|Y|) : \varphi(D) = D \}$$

One can show: $\mathrm{Div}^+\left({|Y|}/{\varphi^{\mathbb{Z}}} \right)$ is free monoid on ${|Y|}/{\varphi^{\mathbb{Z}}}$ via

$$\begin{aligned} {|Y|}/{\varphi^{\mathbb{Z}}} &\hookrightarrow \mathrm{Div}^+\left({|Y|}/{\varphi^{\mathbb{Z}}} \right) \\ y \bmod \varphi^{\mathbb{Z}} &\longmapsto \sum_{n \in \mathbb{Z}} \varphi^n(y). \end{aligned}$$

Def 7: For $f \in B_I \setminus \{0\}$, define

$$\mathrm{div}(f) := \sum_{y \in |Y_I|} \mathrm{ord}_y(f)\, y$$

where $\mathrm{ord}_y(f)$ is the image of f under

$$B_I \longrightarrow B^+_{dR,y} \xrightarrow[\text{val}]{\xi_y\text{-adic}} \mathbb{N}.$$

By Thm 2, $\mathrm{div}(f) \in \mathrm{Div}^+(|Y_I|)$.

Prop 8:

① For $f, g \in B_I \setminus \{0\}$

$$f \in g B_I \iff \mathrm{div}(f) \geq \mathrm{div}(g)$$

② The morph of monoids

$$\mathrm{div}: (B_I \setminus \{0\}) / B_I^{\times} \longrightarrow \mathrm{Div}^+(|X_I|).$$

is injective, and bijective if I is cpct.

Pf: If I cpct both ① and ② OK by (pf of) Thm 2.

For general I,

① Follows from $B_I \cong \varprojlim_{\substack{J \subseteq I \\ \text{cpct}}} B_J$

② Follows from ①, using that B_I is integral $\square$

Will see the desired property of P from the following Thm.

Thm 9: The mph

$$\operatorname{div}: \bigcup_{d \geq 0} (P^d \setminus \{0\}) / E^\times \longrightarrow \operatorname{Div}^+(|Y| / \varphi^{\mathbb{Z}})$$

is an isom of monoids.

Pf:

Well-def: $x \in P^d \setminus \{0\}$

$$\varphi(\operatorname{div}(x)) = \sum_{y \in |Y|} \operatorname{ord}_y(x)\, \varphi(y) = \sum_{\varphi(y) \in |Y|} \operatorname{ord}_{\varphi(y)}(\varphi(x))\, \varphi(y)$$

$$= \sum_{y \in |Y|} \operatorname{ord}_y(\varphi(x))\, y = \operatorname{div}(\varphi(x))$$

$$= \operatorname{div}(\pi^d) + \operatorname{div}(x) = \operatorname{div}(x)$$

π invertible in B

inj: Let $x \in P^d \setminus \{0\}$ and $x' \in P^{d'} \setminus \{0\}$ s.t.

$\mathrm{div}(x) = \mathrm{div}(x')$.

WLOG: $d' \geq d$, by Prop 8: $x = x'u$ for some

$u \in B^\times$.

$$\Rightarrow u \in B^{\varphi = \pi^{d-d'}} = \begin{cases} E & \text{if } d = d' \\ 0 & \text{else} \end{cases}$$

"Newton polygon arguments"

$\rightsquigarrow$ Negative case should really follow from B^+ case, 0 case a bit harder...

surj: STS: $\forall y \in |Y| \; \exists t_y \in P^1$ s.t.

$$\mathrm{div}(t_y) = \sum_{n \in \mathbb{Z}} \varphi^n(y) = \sum_{n \geq 0} \varphi^n(y) + \sum_{n < 0} \varphi^n(y)$$

Let $y \in |Y|$ and write $\varepsilon_y = \pi - [a]$ for some $a \in \mathfrak{m}_F \setminus \{0\}$. Consider

$$\Pi^+(\varepsilon_y) := \prod_{n \geq 0} \frac{\varphi^n(\varepsilon_y)}{\pi}$$

$$= \prod_{n \geq 0} \left(1 - \frac{[a^{q^n}]}{\pi}\right) \in B$$

as $\lim_{n \to \infty} [a^{q^n}] = 0$.

We have

$$\pi \Pi^+(\varepsilon_y) = \varepsilon_y \, \varphi(\Pi^+(\varepsilon_y)) \text{ and}$$

$$\mathrm{div}(\Pi^+(\varepsilon_y)) = \sum_{n \geq 0} \mathrm{div}\left(\frac{\varphi^n(\varepsilon_y)}{\pi}\right) = \sum_{n \geq 0} \varphi^n(y)$$

Remains to find preimage $\Pi^{-}(\varepsilon_y)$ of $\sum_{n<0} \varphi^n(y)$

Candidate

<u>Warning</u>: $\prod_{n<0} \frac{\varphi^n(\varepsilon_y)}{\pi}$ does not exist in B!

<u>Claim</u>: $\exists\, \Pi^{-}(\varepsilon_y) \in B^b$ unique up to mult with $E^\times$ s.t. $\varphi(\Pi^{-}(\varepsilon_y)) = \varepsilon_y \Pi^{-}(\varepsilon_y)$

<u>Sketch pf of claim</u>: We really use that F is alg closed here!

π-complete

Construct $\Pi^{-}(\varepsilon_y)$ as the limit in A_{inf} of a sequence x_n converging wrt π-adic top s.t. $\varphi(x_n) \equiv \varepsilon_y x_n \mod \pi^n$ and $x_n \equiv x_{n+1} \mod \pi^n$ for any $n \geq 1$

Note, $t_y := \pi^+(\varepsilon_y)\,\pi^-(\varepsilon_y) \in P'$ and

$$\varphi(\mathrm{div}(\pi^-(\varepsilon_y))) = \underset{\substack{\| \\ y}}{\mathrm{div}(\varepsilon_y)} + \underset{\substack{\| \\ D}}{\mathrm{div}(\pi^-(\varepsilon_y))}$$

$$\varphi^2(D) = y + \varphi(y) + D$$

$$\vdots$$

$$\varphi^n(D) = y + \dots + \varphi^{n-1}(y) + D$$

$$\Leftrightarrow \quad D = \varphi^{-1}(y) + \dots + \varphi^{-n}(y) + \varphi^{-n}(D)$$

As $\mathrm{supp}(D) \subseteq |Y| = \bigcup_{\substack{I \subseteq (0,\infty) \\ \text{cpct}}} |Y_I|$, $\exists \rho > 0$ s.t.

$$\operatorname{supp}(D) \subseteq |Y_{(0,\rho]}|$$

$$\Rightarrow \operatorname{supp}(\varphi^{-n}(D)) \subseteq |Y_{(0,\frac{1}{q^n}\rho]}|$$

So, for any cpct $I \subseteq (0,\infty)$ $\exists\, n \gg 0$ st

$$\cdot \operatorname{supp}(\varphi^{-n}(D))|_{|Y_I|} = 0$$

$$\cdot \operatorname{supp}(\varphi^{-n}(y))|_{|Y_I|} = 0$$

$\Rightarrow$ $D = \sum_{n<0} \varphi^n(y)$ in $\mathrm{Div}^+(|Y_I|)$ for any cpct I.

$\Rightarrow \quad D = \sum_{n<0} \varphi^n(y)$ in $\mathrm{Div}^+(|Y|)$ $\square$

Rem: Newton polygon arguments show that the ring from Thm 9 is even graded!

Cor 10: The alg P is graded factorial with irred elements of degree 1.
In particular, if $d \geq 1$ and $x \in P^d$ then there exist $t_1, \dots, t_d \in P'$ st $x = t_1 \cdots t_d$.

If there is time:

We used the following in the pf of Thm 9, which we said can be proved with Newton polygon arguments

Prop: For $d \leq 0$

$$P^d \cong \begin{cases} E & \text{if } d = 0 \\ 0 & \text{else} \end{cases}$$

Recall the analogous statement for B^b eigenspaces, seen last time

Prop: For $d \in \mathbb{Z}$

$$(B^b)^{\varphi = \pi^d} = \begin{cases} E & \text{if } d = 0 \\ 0 & \text{else} \end{cases}$$