Structure of finite reductive groups

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Introduction

The finite reductive groups, or finite groups of Lie type, feature prominently in the classification of finite simple groups, providing almost all of the infinite families. The study of finite groups of Lie type blends the study of linear algebraic groups over (algebraic closures of) finite fields and group theory. The geometric aspects of the theory provide a powerful tools to approach their structure. Our goal is to introduce linear algebraic groups, with a focus on their structure and their subgroups.

Plan:
- linear algebraic groups and basic properties
- structure of reductive linear algebraic groups and special subgroups
- classification of semisimple linear algebraic groups
- finite reductive groups and duality

Notation: Fix k an algebraically closed field. By variety we shall always mean a classical variety over k.

1 Linear Algebraic Groups

1.1 Definition and examples

Definition 1.1. A **linear algebraic group** is an affine variety G equipped with a group structure such that the structure maps

$$\mu\colon G \times G \to G, \qquad (g,h) \mapsto gh \qquad \text{and} \qquad \iota\colon G \to G, \qquad g \mapsto g^{-1}$$

are morphisms of varieties.

A **morphism of linear algebraic groups** is a morphism of groups $\varphi\colon G_1 \to G_2$ which is also a morphism of affine varieties. ⌟

Example 1.2. (i) The **additive group** is $\mathbb{G}_a = (k, +)$ understood as the affine variety k. Its coordinate ring is $k[\mathbb{G}_a] \cong k[T]$.

(ii) The **multiplicative group** is $\mathbb{G}_m = (k^*, \cdot)$ endowed with the subspace topology $k^* \subset k$. To see that $\mathbb{G}_m$ is an affine variety, note that we can identify $\mathbb{G}_m$ with the set $\{(x,y) \in k^2 \mid xy = 1\}$ equipped with coordinate-wise multiplication. Since this is an affine variety in k^2, we see that $\mathbb{G}_m$ is a linear algebraic group. Its coordinate ring is $k[\mathbb{G}_m] \cong k[T^{\pm 1}]$.

(iii) The **general linear group** is $\mathrm{GL}_n = \{A \in \mathrm{M}_n(k) \mid \det A \neq 0\}$ endowed with the subspace topology $\mathrm{GL}_n \subset k^{n^2}$. As for $\mathbb{G}_m$, we can identify GL_n with the set $\{(A,y) \in k^{n^2} \times k \mid (\det A)y = 1\}$, where we consider the matrix product for the factor k^{n^2} and multiplication for the factor k. Since the determinant of a matrix is a polynomial expression in its entries, this shows that GL_n is an affine variety in $k^{n^2} \times k$, hence a linear algebraic group. Its coordinate ring is the localisation at the determinant polynomial $k[\mathrm{GL}_n] \cong k[T_{ij} \mid 1 \leq i, j \leq n]_{\det(T_{ij})}$.

(iv) The **special linear group** is the closed subgroup $\mathrm{SL}_n = \{A \in \mathrm{GL}_n \mid \det A = 1\}$ of the general linear group. Since closed subvarieties of affine varieties are affine varieties, this shows that SL_n is a linear algebraic group. Its coordinate ring is $k[\mathrm{SL}_n] \cong k[T_{ij} \mid 1 \leq i, j \leq n]/(\det(T_{ij}) - 1)$.

(v) The **symplectic group** Sp_{2n} (respectively, the **orthogonal group** GO_n) is defined as a closed subgroup of GL_{2n} (respectively, GL_n) preserving a specified non-degenerate skew-symmetric (respectively, symmetric) bilinear form. ⌟

By construction, Example 1.2 (iv) and Example 1.2 (v) are closed subgroups of GL_n for some n. Moreover, $\mathbb{G}_m = \mathrm{GL}_1$ and

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

embeds $\mathbb{G}_a$ as a closed subgroup of GL_2. More generally, there is the following.

Theorem 1.3. *Let G be a linear algebraic group. Then G can be (non-uniquely) embedded as a closed subgroup into GL_n for some n.* □

1.2 First properties

1.2.1 Components Let X be a topological space. Recall, that we say that X is **irreducible**, if X cannot be written as $X = X_1 \cup X_2$ for X_1 and X_2 proper closed subsets and that X is **connected**, if X cannot be written as $X = X_1 \sqcup X_2$ for X_1 and X_2 disjoint non-empty closed subsets. Any irreducible space is connected, but the converse is not necessarily true. Moreover, any topological space can be decomposed into its maximal connected subsets, which we call **connected components**, while every affine variety can be written as a union of its maximal irreducible subsets, which we call **irreducible components**.

As an affine variety is a topological space, it admits both connected as well as irreducible components and each irreducible component is contained in exactly one connected component. For a linear algebraic group, it turns out that its set of connected components and its set of irreducible components agree.

Proposition 1.4. *Let G be a linear algebraic group. The irreducible components of G are pairwise disjoint. In particular, they are equal to the connected components of G.*

Proof. Let X and Y be irreducible components of G. Suppose that $X \cap Y \neq \emptyset$ and let $g \in X \cap Y$. Since multiplication by g^{-1} is a homeomorphism, both $g^{-1}X$ and $g^{-1}Y$ are irreducible components of G as well. Thus, without loss of generality, we may assume that $1 \in X \cap Y$. Since X and Y are irreducible in G, observe that $X \times Y$ is irreducible in $G \times G$. Thus, $\mu(X \times Y) = XY$ is irreducible in G. But $X, Y \subseteq XY$ since $1 \in X \cap Y$. As X and Y were assumed to be irreducible components, hence maximal irreducible subset, we conclude that $X = XY = Y$, as claimed.

Each irreducible component of G is contained in a unique connected component of G and the irreducible components are pairwise disjoint by the above. As each irreducible component of G is a connected subset of G, we conclude that each connected component of G contains exactly one irreducible component of

G. As the irreducible components and the connected components both cover G, we conclude that each connected component is equal to the unique irreducible component it contains. □

By Proposition 1.4 we may unambiguously refer to either irreducible components or connected components of G simply as **components of G**.

Definition 1.5. Let G be a linear algebraic group. The component of G containing the identity is called the **identity component of G** and denoted by G°. ⌟

Proposition 1.6. *Let G be a linear algebraic group. Then G° is a closed normal subgroup of G of finite index and any closed subgroup of G of finite index contains G°.* □

Example 1.7. (i) The linear algebraic groups $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n and SL_n are connected.

(ii) The orthogonal groups GO_n are disconnected for $n \geq 2$. The identity component of GO_n is the **special orthogonal group** $\mathrm{SO}_n = \mathrm{GO}_n^\circ$. ⌟

1.2.2 Abstract Jordan decomposition Let V be a finite dimensional k-vector space. For every endomorphism $a \in \mathrm{End}(V)$, there exists an **additive Jordan decomposition**, meaning there are unique $s, n \in \mathrm{End}(V)$ such that s is **semisimple** (i.e. diagonalisable), n is is nilpotent, $a = s + n$ and $sn = ns$. If $a \in \mathrm{GL}(V)$ and given such an additive Jordan decomposition $a = s + n$, by putting $u = 1 + s^{-1}n$ (note that s is invertible if a is), we produce a **multiplicative Jordan decomposition**, meaning $a = su = us$ with u **unipotent** (i.e. $1 - u$ is nilpotent).

This extends to linear algebraic groups as follows.

Proposition 1.8. *Let G be a linear algebraic group.*

(i) For every embedding ρ of G as a closed subgroup of GL_n for some n and for every $g \in G$, there are unique $g_s, g_u \in G$ such that $\rho(g_s)$ is semisimple, $\rho(g_u)$ is unipotent and $g = g_s g_u = g_u g_s$.

(ii) The decomposition $g = g_s g_u = g_u g_s$ is independent of the chosen embedding. □

For $g \in G$ an element of a linear algebraic group, the decomposition of Proposition 1.8 is called the **(abstract) Jordan decomposition of g**. We say that g is **unipotent** (respectively **semisimple**) if $g = g_u$ (respectively $g = g_s$) and write $\boldsymbol{G_u}$ (respectively G_s) for the subset of unipotent (respectively semisimple) elements of G. A linear algebraic group G is **unipotent** if $G = G_u$

Remark 1.9. The definition of a semisimple linear algebraic group G is *not* $G = G_s$. ⌟

Example 1.10. The linear algebraic group $\mathbb{G}_a$ is unipotent while the linear algebraic group $\mathbb{G}_m$ consists only of semisimple elements. ⌟

2 Reductive and Semisimple Linear Algebraic Groups

2.1 Definition and examples

We recall some concepts from group theory. For a group G, define $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. The **derived series of G** is

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \dots$$

and we say that G is **solvable** if $G^{(i)} = 1$ for some i. If G is a linear algebraic group, then $G^{(1)}$ is closed normal subgroup of G, which is connected if G is connected. In particular, the group-theoretic notion of solvability translates well to the setting of linear algebraic groups.

Definition 2.1. Let G be a linear algebraic group. The maximal closed connected solvable normal subgroup of G is called the **radical of G**, denoted $\mathbf{R}(G)$. The maximal closed connected unipotent normal subgroup of G is called the **unipotent radical of G**, denoted $\mathbf{R}_u(G)$. ⌟

Definition 2.2. Let G be a linear algebraic group. We say that G is **semisimple** if G is connected and $\mathrm{R}(G) = 1$. We say that G is **reductive** if $\mathrm{R}_u(G) = 1$. ⌟

The structure of connected reductive and semisimple linear algebraic groups is more traceable than the structure of arbitrary linear algebraic groups. We record the following results to give a first impression. We say that a non-trivial linear algebraic group is **simple** if it has no non-trivial proper closed connected normal subgroups.

Proposition 2.3. *Let G be a connected reductive group.*

(i) The radical of G is $\mathrm{R}(G) = \mathrm{Z}(G)^\circ$ and the closed connected subgroup $[G,G]$ is semisimple.

(ii) There is a multiplicative decomposition $G = [G,G]\mathrm{R}(G) = [G,G]\mathrm{Z}(G)^\circ$.

(iii) If G is semisimple, then there is a decomposition $G = G_1 \cdots G_r$ which each G_i a simple linear algebraic group. □

Example 2.4. The linear algebraic group GL_n is reductive with radical $\mathrm{R}(\mathrm{GL}_n) = \mathrm{Z}(\mathrm{GL}_n)^\circ \cong \mathbb{G}_m$ and $[G,G] = \mathrm{SL}_n$. In particular, the linear algebraic group SL_n is semisimple. ⌟

2.2 Special subgroups

To get a better handle on the structure of general linear algebraic groups, we consider classes of well-behaved subgroups. The case of GL_n serves as a guiding example.

2.2.1 Maximal Tori A linear algebraic group T is called a **torus** if it is isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

Definition 2.5. Let G be a linear algebraic group G. A **maximal torus** is a closed subgroup $T \leq G$ which is a torus and maximal with respect to inclusion. ⌟

Proposition 2.6. *Let G be a linear algebraic group.*

(i) If G is connected reductive, then $\mathrm{R}(G) = \mathrm{Z}(G)^\circ$ is a torus.

(ii) If G is connected solvable, then all maximal tori of G are conjugate and if T is any maximal torus of G, then $G = G_u \rtimes T$ and $\mathrm{N}_G(T) = \mathrm{C}_G(T)$.

(iii) If G is connected reductive and T is a maximal torus of G, then $\mathrm{C}_G(T) = T$.

Example 2.7. Let $G = \mathrm{GL}_n$. A maximal torus is given by D_n, the closed subgroup of diagonal matrices. ⌟

2.2.2 Borel and parabolic subgroups For some increased flexibility, we not only consider maximal tori, but, more generally, maximal solvable subgroups. Note that solvability is, for instance, needed in Proposition 2.6 (ii) to ensure that maximal tori are well-behaved.

Definition 2.8. Let G be a linear algebraic group.

(i) A **Borel subgroup of G** is a maximal closed connected solvable subgroup of G.

(ii) A **parabolic subgroup of G** is any closed subgroup of G containing a Borel subgroup B of G. ⌟

As Borel subgroups are connected by assumption, they are always contained in G°. Hence, the Borel subgroups of G and G° agree and we typically will assume that G is connected whenever we consider Borel (and parabolic) subgroups. Moreover, note that any linear algebraic group G admits a Borel subgroup: take any closed connected solvable subgroup of G of maximal dimension.

The class of parabolic subgroups is a natural enlargement of the class of Borel subgroups in more than one sense. To see this, let us consider, more generally, a closed subgroup H of G. There is a relative version of Theorem 1.3, which states that G can be embedded in some GL_n such that H stabilises a one-dimensional subspace. This way, we can identify the coset space G/H with the G-orbit of a point in $\mathbb{P}^{n-1}$. Hence, G/H acquires the structure of quasi-projective variety. It can be shown that this structure is independent of the choice of embedding, allowing us to view G/H as a quasi-projective variety.

Proposition 2.9. *Let P be a closed subgroup of a connected linear algebraic group. Then P is parabolic if and only if G/P is a projective variety.*

Remark 2.10. If N is a closed normal subgroup of a linear algebraic group G, then G/N is affine, hence again a linear algebraic group. ⌟

Example 2.11. Let $G = \mathrm{GL}_n$. A Borel subgroup is given by T_n, the closed subgroup of upper triangular matrices. Observe that $D_n \leq T_n$ and the decomposition of Proposition 2.6 (ii) is $T_n = U_n \rtimes D_n$ with U_n the closed subgroup of unipotent upper triangular matrices. ⌟

2.2.3 Levi subgroups If G is a (connected) linear algebraic group and B is Borel subgroup, then $B = \mathrm{R}_u(B) \rtimes T$ for T any maximal torus of B (hence of G) by Proposition 2.6 (ii). We want an analogue of this result for parabolic subgroups of G. As parabolic subgroups are not necessarily solvable, we cannot just apply Proposition 2.6 (ii).

Definition 2.12. A connected linear algebraic group P has a **Levi decomposition**, if there is a closed subgroup L such that $P = \mathrm{R}_u(P) \rtimes L$. The group L is called a **Levi subgroup of P**, or a **Levi complement**.⌟

Proposition 2.13. *Let G be a connected reductive linear algebraic group. Fix a maximal torus T and let P be a parabolic subgroup of G containing T.*

(i) There is a unique Levi subgroup of P containing T.

(ii) Any two Levi subgroups of P are conjugate by a unique element of $\mathrm{R}_u(P)$.

Proposition 2.14. *Let G be a connected reductive linear algebraic group.*

(i) Let L be a Levi subgroup of a parabolic subgroup of G. Then $L = \mathrm{C}_G(\mathrm{Z}(L)^\circ)$.

(ii) Let S be a torus of G. Then $\mathrm{C}_G(S)$ is the Levi subgroup of some parabolic subgroup of G.

We discuss the ideas going into the proof of Propositions 2.13 and 2.14 in Section 3.3, after our discussion of root systems.

Example 2.15. Consider positive integers $n_1, \ldots, n_r$ such that $n_1 + \cdots + n_r = n$. The closed subgroup $P_{n_1,\ldots,n_r}$ of GL_n of block upper triangular matrices with blocks along the diagonal of size $n_1 \times n_1, \ldots, n_r \times n_r$ is a parabolic subgroup containing the Borel subgroup T_n of upper triangular matrices. For D_n the maximal torus of diagonal matrices, the Levi factor of $P_{n_1,\ldots,n_r}$ is given by the closed subgroup $D_{n_1,\ldots,n_r}$ of block diagonal matrices with blocks of size $n_1 \times n_1, \ldots, n_r \times n_r$. The unipotent radical of $P_{n_1,\ldots,n_r}$ is given by the subgroup of those block upper triangular matrices for which the diagonal blocks are the identity matrices of appropriate size. ⌟

3 Classification of Semisimple Linear Algebraic Groups

3.1 Root systems and root data

Semisimple linear algebraic groups can be elegantly classified using the notion of a root datum. This is an enhancement of the notion of a root system, which appears in the classification of semisimple Lie algebras. A primary advantage is, that root systems and root data are incredibly explicit combinatorial objects.

3.1.1 Root systems Let E be an **euclidean vector space**, meaning a finite dimensional real vector space endowed with a positive definite symmetric bilinear form $(-,-)$. Given $\alpha, \beta \in E$, let $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ where $\langle \beta, \alpha \rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$. This is the **reflection determined by α**, leaving the hyperplane $\{\beta \in E \mid (\beta, \alpha) = 0\}$ pointwise fixed and sending α to $-\alpha$.

Definition 3.1. A subset Φ of an euclidean vector space E is called an **(abstract) root system** if

R1) Φ is finite, spans E and $0 \notin E$;

R2) if $\alpha \in \Phi$ and $c \in \mathbb{R}$ such that $c\alpha \in \Phi$, then $c = \pm 1$;

R3) for each $\alpha \in \Phi$, the reflection s_α stabilises Φ;

R4) if $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Two root systems Φ in E and Φ' in E' are **isomorphic** if there is a linear isomorphism $f \colon E \to E'$ such that $\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in \Phi$. ⌟

Definition 3.2. Let Φ be a root system in E. The subgroup generated by reflections $W = \langle s_\alpha \mid \alpha \in \Phi \rangle \leq \mathrm{GL}(E)$ is called the **Weyl group of Φ**. ⌟

Remark 3.3. Since the reflections s_α, $\alpha \in \Phi$, permute the finite set Φ, the Weyl group is finite. Moreover, isomorphic root systems have canonically isomorphic Weyl groups. ⌟

The Weyl group is an invaluable tool when studying root systems. In particular, they are used in the classification of root systems. We say that root system Φ in E is **decomposable** if there are proper subsets Φ_1, Φ_2 such that $(\alpha_1, \alpha_2) = 0$ for all $\alpha_1 \in \Phi_1$ and $\alpha_2 \in \Phi_2$. Otherwise, we say that Φ is **indecomposable**.

Theorem 3.4. *Let Φ be an indecomposable root system. Then, up to isomorphism, Φ corresponds to exactly one of the following **types**:*

$$A_n \ (n \geq 1), \quad B_n \ (n \geq 2), \quad C_n \ (n \geq 3), \quad D_n \ (n \geq 4), \quad E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2$$

The types in Theorem 3.4 correspond to types of **Dynkin diagrams** (see [MT11, Table 9.1]). The Dynkin diagrams are graphs which can be extracted from the combinatorics of the root system.

3.1.2 Root data Given a root system Φ in E and a root $\alpha \in \Phi$, its **coroot** is $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. The collection of coroots assembles into the **dual root system** $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$, which is again a root systems. For the study of semisimple and reductive linear algebraic groups, we shall not only include the information about the coroots, but also consider an integral version of root systems.

Definition 3.5. A quadruple $(X, \Phi, Y, \Phi^\vee)$ is called a **root datum** if

RD1) $X \cong \mathbb{Z}^n \cong Y$ with a perfect pairing $\langle -, - \rangle \colon X \times Y \to \mathbb{Z}$;

RD2) $\Phi \subseteq X$ and $\Phi^\vee \subseteq Y$ are abstract root systems in $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, respectively;

RD3) there exists a bijection $(-)^\vee \colon \Phi \to \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$;

RD4) the reflection s_α, $\alpha \in \Phi$, is given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \qquad \text{for all } x \in X$$

and the reflection $s_{\alpha^\vee}$, $\alpha^\vee \in \Phi^\vee$, is given by

$$s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee \qquad \text{for all } y \in Y\,.$$

Two root data $(X, \Phi, Y, \Phi^\vee)$ and $(X', \Phi', Y', \Phi'^\vee)$ are **isomorphic** if there is an isomorphism $f \colon X' \to X$ that maps Φ' to Φ and for which the dual isomorphism $f^\vee \colon Y \to Y'$ maps $f(\beta)^\vee$ to $\beta^\vee$ for each $\beta \in \Phi'$. ⌟

Remark 3.6. If $(X, \Phi, Y, \Phi^\vee)$ is a root datum, then so is $(Y, \Phi^\vee, X, \Phi)$. We call this the **dual root datum**.⌟

By definition, each root datum gives rise to two root systems, which are dual to each other. In general, a root datum is not determined by the associated root system(s). However, it is almost if the root datum comes from a linear algebraic group (see Section 3.2). The relevant condition on the root datum can be introduced without this context, though.

Let $(X, \Phi, Y, \Phi^\vee)$ be a root datum. Denote $\Omega = \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$ the $\mathbb{Z}$-linear dual of the coroot lattice. The perfect pairing of the root datum then yields a natural homomorphism

$$X \cong \mathrm{Hom}(Y, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) = \Omega$$

which is injective. Thus, we may view $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$. We say that the root datum is **semisimple** if $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R} = X \otimes_{\mathbb{Z}} \mathbb{R}$. For a semisimple root datum $(X, \Phi, Y, \Phi^\vee)$, the index $|\Omega : \mathbb{Z}\Phi|$ is finite and the finite group $\Lambda = \Omega/\mathbb{Z}\Phi$ is called the **fundamental group of $(X, \Phi, Y, \Phi^\vee)$**. It does not depend on X.

Proposition 3.7. *Let Φ be a root system. A semisimple root datum with Φ as root system is completely determined by the associated fundamental group.* □

3.2 Root Data for Linear Algebraic Groups

We would like to use the notions introduced in Section 3.1 in the study of reductive linear algebraic group. A thorough treatment is beyond the scope of this note, so we will just highlight a few aspects.

Let start by sketching how to attach a root system to a reductive linear algebraic group G. Fix a maximal torus $T \leq G$ and let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G, that is, the tangent space of G at the identity element. Note that G acts on itself by conjugation. This induces a morphism $\mathrm{Int}\colon G \to \mathrm{End}(G)$ whose derivative yields a morphism $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, called the **adjoint representation**. The image of the maximal torus T under Ad is a set of commuting semisimple elements of $\mathrm{GL}(\mathfrak{g})$, hence simultaneously diagonalisable. Denote by $\boldsymbol{X(T)} \coloneqq \mathrm{Hom}(T, \mathbb{G}_m)$ the **character group of $\boldsymbol{T}$**. For each $\chi \in X(T)$, let

$$\mathfrak{g}_\chi \coloneqq \{x \in \mathfrak{g} \,|\, (\mathrm{Ad}\, t)(x) = \chi(t)x \text{ for all } t \in T\}.$$

Then $\mathfrak{g} = \bigoplus_{\chi \in X(T)} \mathfrak{g}_\chi$. We call

$$\boldsymbol{\Phi(G)} \coloneqq \{\chi \in X(T) \,|\, \chi \neq 0,\ \mathfrak{g}_\chi \neq 0\}$$

the set of **roots of $\boldsymbol{G}$ with respect to $\boldsymbol{T}$**. Denote by $\boldsymbol{Y(T)} \coloneqq \mathrm{Hom}(\mathbb{G}_m, T)$ the **cocharacter group of $\boldsymbol{T}$**. It can be shown, that for each $\alpha \in \Phi(G)$, there is a unique $\alpha^\vee \in Y(T)$. Let $\boldsymbol{\Phi(G)^\vee} = \{\alpha^\vee \,|\, \alpha \in \Phi(G)\}$.

Proposition 3.8. *Let G be a reductive linear algebraic group. Then $(\Phi(G), X(T), \Phi(G)^\vee, Y(T))$ is a root datum. If G is semisimple, then $\Phi(G) \otimes_{\mathbb{Z}} \mathbb{R} = \Phi(G) \otimes_{\mathbb{Z}} X(T)$.*

With the notion of a root datum attached to a reductive linear algebraic group in hand, we can define when two such groups are dual to each other. Shortly, the dual of a reductive linear algebraic group is the reductive linear algebraic group whose root datum is the dual root datum.

Definition 3.9. Let G and G^* be connected reductive linear algebraic groups. We say that G and G^* are **dual**, or **in duality**, if there are maximal tori $T \leq G$ and $T^* \leq G^*$ so that for the associated root data $(X(T), \Phi(G), Y(T), \Phi(G)^\vee)$ and $(X(T^*), \Phi(G^*), Y(T^*), \Phi(G^*)^\vee)$ there is an isomorphism $\delta\colon X(T) \to Y(T)$ satisfying $\delta(\Phi(G)) = \Phi(G^*)^\vee$ and

$$\langle \lambda, \alpha^\vee \rangle = \langle \alpha^*, \delta(\lambda) \rangle \qquad \text{for all } \lambda \in X(T) \text{ and } \alpha \in \Phi(G)$$

where $\alpha^* \in \Phi(G^*)$ is defined by $\delta(\alpha) = (\alpha^*)^\vee$ ⌟

Finally, let us mention the classification theorem of Chevalley for semisimple linear algebraic groups.

Theorem 3.10. *Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum, there exists a semisimple linear algebraic group which realises it. A semisimple linear algebraic group is simple if and only if its root system is indecomposable.*

3.3 Application: Parabolic Subgroups and Levi Decomposition

In this section we sketch how one approaches the proof of results such as Propositions 2.13 and 2.14 using the associated root system. Fix G a reductive linear algebraic group, let T be a maximal torus of G and $T \leq B$ a Borel subgroup of G. Let $\Phi(G)$ be the root system of G with respect to T.

We let $W = \mathrm{N}_G(T)/T$ denote the **Weyl group of G**, which is canonically isomorphic to the Weyl group of the root system $\Phi(G)$. There is a **basis** of $\Phi(G)$, which is to say, a subset $\Delta \subset \Phi(G)$ of roots such that the elements of Δ are a linear basis and each element of $\Phi(G)$ can be written as a $\mathbb{Z}$-linear combination in Δ with either all positive or negative coefficients. We call the elements of Δ the **simple roots**. Write $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the **simple reflections**. It turns out that S generates W. For $I \subseteq S$, we call a subgroup $W_I = \langle s \in I \rangle$ a **standard parabolic subgroup of W** and any subgroup conjugate to such a subgroup a **parabolic subgroup of W**.

The subgroups B and $N = \mathrm{N}_G(T)$ together with the associated Weyl group W define a ***BN*-pair**. In particular, there is a decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B$$

for $\dot{w} \in N$ any preimage of $w \in W$. This is the **Bruhat decomposition of G**. If $I \subset S$ and W_I is a the associated standard parabolic subgroup of W, we call

$$P_I := BW_IB = \bigsqcup_{w \in W_I} B\dot{w}B$$

a **standard parabolic subgroup of G**. These subgroups turn out to be parabolic subgroups of G in the sense of Definition 2.8 (ii).

Proposition 3.11. *Let $P \leq G$ be a parabolic subgroup of G. Then P is conjugate to P_I for some $I \subseteq S$ and if $P_I = P_J$, then $I = J$.* □

With Proposition 3.11, the proofs of Propositions 2.13 and 2.14 turn into a detailed analysis of the root systems attached. For instance, given a standard parabolic subgroup P_I of G, its Levi complement L_I as well as its unipotent radical $\mathrm{R}_u(P_I)$ can be explicitly expressed in terms of the root system of G. This makes the structure of P_I completely transparent.

Let us not delve into the proofs, but instead let us highlight the utility of this approach with an example.

Example 3.12. Let $n \geq 2$ and write $n = n_1 + n_2$ with $n_1, n_2 \geq 1$. Consider the linear algebraic group Sp_{2n}. One can embed $\mathrm{Sp}_{2n_1} \times \mathrm{Sp}_{2n_2}$ into Sp_{2n}. However, contrary to the GL_n-case, this embedding does *not* define a parabolic subgroup of Sp_{2n}. Indeed, the root systems of Sp_{2n} contains a single double vertex. As the root systems associated to parabolic subgroups are obtained by deleting vertices and edges from the root system of the ambient reductive group, this is impossible. ⌟

4 Finite Reductive Groups

Fix k to be the algebraic closure of a finite field of characteristic p. Suppose that V is an affine variety over k which is defined by a set of polynomials $I \leq \mathbb{F}_{p^a}[T_1, \ldots, T_n]$. Then we say that **$X$ is defined over $\mathbb{F}_{p^a}$**.

The Frobenius automorphism

$$F_{p^a}\colon k \longrightarrow k, \qquad x \longmapsto x^{p^a}$$

acts naturally on $k[T_1,\dots,T_n]$, leaving I invariant. Thus, F_{p^a} acts on V, the set of common zeroes of I, by

$$F_{p^a}\colon F_{p^a}\colon k \longrightarrow k, \qquad (v_1,\dots,v_n) \longmapsto \left(v_1^{p^a},\dots,v_n^{p^a}\right).$$

We call F_{p^a} the **Frobenius endomorphism of V with respect to the $\mathbb{F}_{p^a}$-structure given by I** and write

$$V^{F_{p^a}} := \{v \in V \mid F_{p^a}(v) = v\}.$$

for the F_{p^a}-fixed points of V.

In the context of linear algebraic groups, we need a slight generalisation of the above.

Definition 4.1. Let G be a linear algebraic group over $\overline{\mathbb{F}_p}$. An endomorphism $F\colon G \to G$ such that for some $m \geq 1$ the power $F^m\colon G \to G$ is the Frobenius morphism with respect to some $\mathbb{F}_{p^a}$-structure on G is called a **Steinberg endomorphism** or **Frobenius root**. We write G^F for the group of F-fixed points of G.⌟

Example 4.2. Let $G = \mathrm{GL}_n$ and view k as the algebraic closure of $\mathbb{F}_q$. For each $a \geq 1$, we have the **standard Frobenius** of GL_n defined by

$$F_{q^a}\colon (a_{i,j})_{i,j} \longmapsto \left(a_{i,j}^{q^a}\right)_{i,j}.$$

An example of a non-trivial Steinberg endomorphism is given by

$$F\colon (a_{i,j})_{i,j} \longmapsto \left(a_{i,j}^{q}\right)_{i,j}^{-\top},$$

composing the standard Frobenius F_q with the morphism sending a matrix to the transpose of its inverse. This morphism satisfies $F^2 = F_{q^2}$. ⌟

From the perspective of finite group theory, Steinberg endomorphisms are precisely what one needs to consider in order to define finite groups as sets of fixed points (at least for simple linear algebraic groups).

Theorem 4.3. *Let G be a simple linear algebraic group and $\sigma\colon G \to G$ and endomorphism of G. Then precisely one of the following holds:*

(i) σ is an automorphism of algebraic groups, or

(ii) the group $G^\sigma := \{g \in G \mid \sigma(g) = g\}$ is finite.

The second case occurs if and only if σ is a Steinberg endomorphism.

We are now ready to make the following central definition.

Definition 4.4. Let G be a reductive linear algebraic group, $F\colon G \to G$ a Steinberg endomorphism. The finite group of fixed points G^F is called a **finite reductive group** or **finite group of Lie type**. ⌟

Example 4.5. Let $G = \mathrm{GL}_n$ and consider the morphisms F_{q^a} and F as in Example 4.2. Clearly,

$$\mathrm{GL}_n(q^a) := (\mathrm{GL}_n)^{F_{q^a}} = \mathrm{GL}_n(\mathbb{F}_{q^a}).$$

We call $\mathrm{GU}_n(q) := (\mathrm{GL}_n)^F$ the **general unitary group** and $\mathrm{SU}_n(q) := (\mathrm{SL}_n)^F$ the **special unitary group**. Despite the suggestive notation, these groups *cannot* be obtained as subgroups of $\mathrm{GL}_n(q)$. ⌟

We close the section by discussing duality for finite reductive groups.

Definition 4.6. Consider two pairs (G, F) and G^*, F^* where G, G^* are connected reductive linear algebraic groups and $F\colon G \to G$, $F^*\colon G^* \to G^*$ are Steinberg endomorphisms. We say that (G, F) and (G^*, F^*) are **dual**, or **in duality**, if there are maximal tori $T \leq G$ and $T^* \leq G^*$ such that G and G^* are dual in the sense of Definition 3.9 and such that the duality map $\delta\colon X \to Y^*$ furthermore satisfies

$$\delta\big(\lambda \circ F|_T\big) = F^*\big|_{T^*} \circ \delta(\lambda) \qquad \text{for all } \lambda \in X\,.$$

⌟

Example 4.7. (i) The finite reductive group $\mathrm{GL}_n(q)$ is self-dual

(ii) The finite semisimple groups $\mathrm{SL}_n(q)$ and $\mathrm{PGL}_n(q)$ are dual.

(iii) The finite semisimple groups $\mathrm{SO}_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$ are dual. ⌟

References

These notes were prepared to accompany my talk in the Wuppertal Forschungsseminar on the topic "Deligne–Lusztig theory and block theory of finite reductive groups". You can find the program here.

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