Seminar: Condensed Mathematics

To the dismay of undergraduates and researchers alike, a continuous bijection between topological spaces need not be a homeomorphism. As a consequence, categories of objects carrying some topology (e.g. topological abelian groups, topological vector spaces over some topological field, etc.) are usually not abelian, and several subtleties arise when one attempts to employ tools from homological algebra or derived category theory.

Condensed sets were introduced by Clausen–Scholze in [6] to provide a categorical setting in which this tension between topology and algebra is overcome: condensed sets contain all 'reasonable' topological spaces as a full subcategory, and condensed abelian groups form a nice Grothendieck abelian category.

The notion of a solid abelian group, and more generally of a solid module over some analytic ring, is roughly playing the role of completeness, together with a natural notion of completed tensor product. As the derived category of solid A-modules satisfies suitable descent properties, this provides a very clean definition of (derived) quasi-coherent $\mathcal{O}$-modules in analytic geometry. As an application, we can discuss six-functor formalisms and coherent duality for schemes (by viewing schemes as discrete adic spaces).

We follow [6] quite closely – for $n > 1$, Talk n covers the material of lecture $n-1$ in [6]. There have been many seminars doing this over the last couple of years, so one may also find corresponding notes online, e.g. [2], [5].

Part I: Basic properties and compatibilities

We introduce condensed sets, condensed abelian groups, etc., and show that these are well-behaved from a categorical point of view. We then verify that this approach is compatible with classical definitions – e.g. for compact Hausdorff spaces, singular cohomology can be computed as condensed cohomology with integral coefficients.

For these first few talks, additional background material can be found in [3].

Talk 1: Introduction and overview. Introduction and overview, distribution of talks.

Talk 2: Condensed sets. Define condensed sets (condensed abelian groups, etc.). Describe how a topological space gives rise to a condensed set. Mention the set-theoretic issues, and the fix of κ-condensed sets – feel free to omit the details here. An alternative fix is to restrict to metrizable profinite sets to obtain 'light' condensed sets, see [5]. Discuss compactly generated spaces and Stone-Cech compactification. State and prove Theorem 1.7, and point out Example 1.9.

Talk 3: Condensed abelian groups. The main result is Theorem 2.2. Introduce extremally disconnected spaces and use Stone-Cech compactification to show that condensed sets (sheaves on profinite sets) are equivalent to sheaves on extremally disconnected spaces (Proposition 2.7). Discuss the symmetric monoidal structure on Cond(Ab) and its derived category.

Talk 4: Condensed cohomology. Recall the various cohomologies for a compact Hausdorff space S and prove Theorem 3.2. Explain the passage to a simplicial hypercover as a suitable resolution, but we will not need the full theory. State Theorem 3.3, and - time permitting - sketch a proof.

Talk 5: Locally compact abelian groups. Calculate the Ext groups of locally compact abelian groups in the condensed setting and observe that the answers agree with Hoffmann–Spitzweck's earlier work (details on which should not be necessary). Feel free to omit some of the details (like the proof of Theorem 4.5 given in the appendix), but we will need the results from Theorem 4.3 in later talks.

Part II: Solid modules

If A and B are topological abelian groups, $A \otimes_{\mathbb{Z}} B$ is usually quite pathological. It is common to opt for a completed tensor product, but there are usually many different choices, each with its own problems and subtleties. Replacing 'complete' by 'solid', we develop the theory from a condensed perspective, where the solid tensor product provides a well-behaved monoidal functor.

Talk 6: Solid abelian groups. Introduce solid abelian groups. State Theorem 5.8. Prove 5.9 and 5.10 and indicate how they can be used to prove 5.8.

Talk 7: Solid abelian groups II. Finish the proof of Theorem 5.8, and discuss the symmetric monoidal structure via the solid tensor product. Also discuss the examples in 6.4.

Talk 8: Analytic rings. Solid abelian groups are solid modules over $\mathbb{Z}$ – analytic rings are the 'rings' (with extra data) over which we can in general define solid modules. Introduce pre-analytic and analytic rings. Explain examples like $\mathbb{Z}_p$, $(A, \mathbb{Z})$ for A discrete, and the Huber pair $(\mathbb{Q}_p, \mathbb{Z}_p)$.

Talk 9: Solid modules and the lower-shriek functor. Recall the notion of a lower-shriek functor (pushforward with compact support) in geometry. State Theorems 8.1 and 8.2 and discuss the consequences 8.3–8.5. Spell out the key example $A = \mathbb{Z}[T]$ in great detail.

Part III: Solid $\mathcal{O}$-modules in geometry

To any analytic ring A, we can now associate the derived category $D(A_\blacksquare)$ of solid A-modules, and the solid tensor product yields a way to localize. In this way, we can develop a theory of solid $\mathcal{O}$-modules. We do so for discrete adic spaces (in particular, schemes – see [1] for a theory for more general adic spaces), and obtain as an application a very clean proof of coherent duality on schemes due to our construction of the lower-shriek functor.

Talk 10: Discrete adic spaces. Introduce discrete adic spaces, and explain how schemes can be regarded as discrete adic spaces. State Theorem 9.8, and explain the need to work with the ∞-category $\mathrm{D}((A, A^+)_\blacksquare)$.

Talk 11: Globalization. Prove Theorem 9.8. You might want to recall some ideas from ∞-categories and point out where they are hidden in the proof.

Talk 12: Coherent duality. Use the earlier results to define the lower-shriek functor. Prove Theorem 11.1. When discussing six-functor formalisms, you might also want to mention [4, Appendix A.5], which gives a precise definition.

References

[1] G. Andreychev. Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces. arXiv 2105.12591.

[2] D.-C. Cisikinski, C. Scheimbauer, et al. Notes from a seminar on condensed/pyknotic mathematics. `https://cisinski.app.uni-regensburg.de/condensed.html`.

[3] B. le Stum. An introduction to condensed mathematics. `https://perso.univ-rennes1.fr/bernard.le-stum/bernard.le-stum/Enseignement_files/CondensedBook.pdf`.

[4] L. Mann. A p-adic 6-Functor formalism in rigid analytic geometry. arXiv 2206.02022.

[5] J. E. Rodriguez Camargo. Notes on solid geometry. `https://blogs.cuit.columbia.edu/jr4460/files/2024/09/SeminarSolidGeometrynotes2.pdf`.

[6] P. Scholze. Lectures on Condensed Mathematics. Based on joint work with D. Clausen. `https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf`.