ADDENDUM "SOLID ABELIAN GROUPS 2"

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During my talk on Section 6 of Condensed.pdf, I failed to provide a convincing argument that the symmetric monoidal product $\otimes^{L\blacksquare}$ is the left derived functor of $\otimes^{\blacksquare}$. The purpose of this note is to do so.

We must prove that for compact projective objects M and N of Solid, there is an equivalence $M \otimes^{L\blacksquare} N \simeq M \otimes^{\blacksquare} N$; that is, the left-hand side remains concentrated in degree zero. We may reduce to the case of compact projective generators, that is, $M = \mathbb{Z}[S]^{\blacksquare}$ and $N = \mathbb{Z}[T]^{\blacksquare}$ for S and T extremally disconnected.

In other words, we are left to construct an equivalence

$$\mathbb{Z}[S]^{\blacksquare} \otimes^{L\blacksquare} \mathbb{Z}[T]^{\blacksquare} \simeq \mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} \mathbb{Z}[T]^{\blacksquare}.$$

We spell out both sides, starting with the left-hand side. Since e.g. S is extremally disconnected, $\mathbb{Z}[S]^{\blacksquare} \simeq \mathbb{Z}[S]^{L\blacksquare}$ by Theorem 5.8(ii), and so the left-hand side is

$$\mathbb{Z}[T]^{L\blacksquare} \otimes^{L\blacksquare} \mathbb{Z}[T]^{L\blacksquare} \simeq (\mathbb{Z}[S] \otimes^{\mathbb{L}} \mathbb{Z}[T])^{L\blacksquare} \simeq (\mathbb{Z}[S] \otimes \mathbb{Z}[T])^{L\blacksquare} = \mathbb{Z}[S \times T]^{L\blacksquare}$$

with the first equivalence due to monoidality and the second due to e.g. S being extremally disconnected. The right-hand side is $\mathbb{Z}[S \times T]^{\blacksquare}$.

We now apply the Yoneda lemma, and since both objects are in $D(\text{Solid})$ (for the right-hand side we are using Theorem 5.8(ii)), it suffices to test this against solid objects. So let $C \in D(\text{Solid})$ be arbitrary. There are equivalences

$$\text{RHom}(\mathbb{Z}[S \times T]^{L\blacksquare}, C) \xrightarrow{\simeq} \text{RHom}(\mathbb{Z}[S \times T], C) \xleftarrow{\simeq} \text{RHom}(\mathbb{Z}[S \times T]^{\blacksquare}, C).$$

The left-hand equivalence uses that $(-)^{L\blacksquare}$ is the left adjoint of the inclusion of $D(\text{Solid})$ into $D(\text{Cond(Ab)})$ (Theorem 5.8(ii)), while the right-hand equivalence uses the definition of being solid in $D(\text{Cond(Ab)})$ (as well as Theorem 5.8(ii) to identify the category of such objects C with $D(\text{Solid})$).