

RESEARCH SEMINAR ON THE GEOMETRIC SATAKE EQUIVALENCE

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This is a seminar aiming to go through the proof of the geometric Satake equivalence of Ginzburg [6] and Mirković-Vilonen [11].

1. INTRODUCTION

Motivation. Geometric representation theory is a branch of mathematics which aims, among other things, at constructing bridges between categories of interesting representations and categories of sheaves on interesting geometric objects, often with the hope of constructing or realising representations from geometric objects. In situations where the geometry is well-understood, this helps turn hard problems in representation theory into problems of a more geometric nature, where new tools become available in order to obtain a solution. The first explicit instance of such a philosophy is found in the work of Beilinson-Bernstein [2], who proved a conjecture of Kazhdan-Lusztig on the characters of highest weight representations of complex semisimple Lie algebras by turning it into a problem about decompositions of perverse sheaves on flag varieties.

The geometric Satake equivalence is a theorem in a similar spirit; however, it is rather more involved and technical than the aforementioned work of Beilinson-Bernstein. Roughly speaking, it relates the representation category of the Langlands dual $G^\vee$ of a connected reductive group G with a certain category of equivariant perverse sheaves on the affine Grassmannian Gr_G of G . This result (or, rather, its many variants) has now become one of the cornerstones of modern geometric representation theory and has numerous applications, for instance in the local and geometric Langlands program (see e.g. [20], [5, §VI] or [18, §2]) or in the representation theory of algebraic groups in positive characteristic (see [15]). It also gives, as far as I know, the only known explicit construction of the Langlands dual group (i.e. not simply invoking its existence by the classification of reductive groups by root data).

Statement of the result. The geometric Satake equivalence knows various iterations depending on whether one considers affine Grassmannians defined over $\mathbb{C}$ or over fields of positive characteristic, whether one considers groups defined over algebraically closed fields or other (e.g. p -adic) fields, whether one works with complex analytic or ℓ -adic constructible sheaves, etc. While there are technical differences between these different setups, the general strategy to prove this result tends to remain uniformly the same. For this reason, and to keep the technical prerequisites at a reasonable level, we will in this seminar work over $\mathbb{C}$ and consider sheaves over a coefficient field k of characteristic zero.

We now state the result more precisely. Let G be a connected reductive group over $\mathbb{C}$ and write $\mathcal{O} := \mathbb{C}[[t]]$ and $K := \mathbb{C}((t))$. The affine Grassmannian Gr_G of G is a certain ind-scheme whose $\mathbb{C}$ -valued points are given by the quotient $G(K)/G(\mathcal{O})$. We let L^+G denote the group functor $R \mapsto G(R[[t]])$ on $\mathbb{C}$ -algebras, which one can show is represented by a $\mathbb{C}$ -group scheme (not of finite type). The affine Grassmannian further has a certain stratification by L^+G -orbits analogous to the Cartan decomposition in the theory of p -adic groups. The *Satake category* $\mathrm{Sat}_{G,k}$ is defined to be the category of L^+G -equivariant perverse sheaves on Gr_G . This is a certain abelian category, which concretely lives as a subcategory of the derived category $D_c(\mathrm{Gr}_G, k)$ of constructible sheaves of k -vector spaces on Gr_G with respect to the aforementioned stratification. The theorem then goes as follows.

Theorem 1.1 (Geometric Satake). *Let $G_k^\vee$ denote the connected split reductive k -group scheme with root datum dual to that of G .*

- (i) *The category $\mathrm{Sat}_{G,k}$ is equipped with a so-called convolution product, giving it the structure of a symmetric monoidal category.*
- (ii) *There is an equivalence of symmetric monoidal categories*

$$\mathrm{Sat}_{G,k} \longrightarrow \mathrm{Rep}(G_k^\vee)$$

to the category of (finite dimensional) representations of $G_k^\vee$.

The main technical difficulty in proving this theorem lies in the study of the convolution product on $\text{Sat}_{G,k}$, i.e. in constructing a tensor product on perverse sheaves which matches up with the way representations tensor together. Once the first part of the theorem is proved, the main formal input to achieve the second part is the theory of Tannakian reconstruction, which will essentially give for free that $\text{Sat}_{G,k}$ is equivalent to the category of representations of some algebraic group over k . The last hurdle is to then identify that group with $G_k^\vee$.

Finally, as a motivation for the name ‘geometric Satake’, when working with p -adic reductive groups instead of complex reductive groups, the resulting equivalence of categories one obtains between the Satake category and the representations of $G^\vee$ induces an isomorphism at the level of Grothendieck rings. The rings $K_0(\text{Sat}_G)$ and $K_0(\text{Rep}(G^\vee))$ can in that case be identified with certain Hecke algebras, and the resulting isomorphism between them recovers a classical theorem of Satake [17].

2. LIST OF TALKS

There is a wealth of resources on this topic, some of which use rather advanced language as is now standard in the geometric Langlands program. We will aim to keep things approachable and follow primarily [14, 21] for the study of affine Grassmannians and [1] for the proof of the main theorem. In particular, we will not assume prior knowledge on affine Grassmannians or perverse sheaves. One may also find the master’s thesis [13] a useful reference for additional details. The speakers are encouraged to speak to the organiser if they have any questions.

The talks go roughly as follows. There will be a ‘talk 0’ giving an overview of the seminar. Then the first five talks explain who the players in the theorem are: talk 1 is a review of the theory of reductive groups, talks 2-3 introduce and study the affine Grassmannian and talks 4-5 introduce perverse sheaves and the Satake category. To prove the result, one needs the Tannakian formalism covered in talk 6. Then talks 7-11 discuss the proof of the theorem, with talks 7-10 focusing on constructing the symmetric monoidal structure and fiber functor necessary in order to implement the Tannakian formalism and talk 11 focusing on reconstructing the dual group. Finally, the last talk discusses how this result connects to the wider Langlands program. We now give details on what each of the talks should cover.

Talk 0 (14.10): Introduction. Introduction and overview of the seminar.

Talk 1 (28.10): Reductive groups and their representations. This talk recalls the structure theory of (split) reductive groups and their representations. The emphasis should not be on proofs here but it would be good, whenever possible, to illustrate the concepts introduced with examples.¹

First recall the definition of affine algebraic groups over a field k and the notions of tori, unipotent and solvable groups (cf. [10, §§6.e, 12.e, 14.a-b & 16.d]). Then recall the definitions of (split) reductive groups as well as their Borel and parabolic subgroups and their root data (cf. [10, §§17.b, 19.b & 21.c]), and explain that split reductive groups are classified up to isomorphism by their root data via the so-called isogeny and existence theorems (cf. [10, §§23.g]). Using this, define the Langlands dual group. Finally, give an overview of the representation theory of a split reductive group G : the irreducible representations are classified by their highest weights (cf. [9, II.2.4-II.2.7]) and, in characteristic zero, the reductivity of G is characterised by the fact that the category $\text{Rep}(G)$ of finite dimensional G -representations is semisimple (cf. [10, §§22.c]).

Talk 2 (04.11): Affine Grassmannians and loop groups. The aim of this talk is to introduce the affine Grassmannian of a reductive group and show that it is an ind-projective ind-scheme. First recall the formalism of ind-schemes via the functor of points, cf. [14, §1]. Then define the affine Grassmannian for GL_n and prove that it is ind-projective, cf. [21, §§1.1] or [14, Thm 2.2]. Next, define the affine Grassmannian for general algebraic group G and sketch the proof that Gr_G is ind-projective whenever G is reductive, cf. [21, §§1.3]. It may help the speaker to check the functorial description of the classical Grassmannian and also the identification of rank n vector bundles with GL_n -torsors, see e.g. [7, §§8.4 & 11.6] or also [13, Lemma 2.2.3]. Finally, define the loop group LG and positive loop group L^+G , give their basic properties and explain that Gr_G may be identified (as an fpqc sheaf) with the quotient LG/L^+G , cf. [21, §§1.3] or [14, §§3.3].

¹Julian’s talk from last term may also be helpful in preparing this one, see https://az.uni-wuppertal.de/fileadmin/mathe/reine_mathematik/az/Julian_Structure_of_finite_reductive_groups.pdf

Talk 3 (11.11): Geometry of the affine Grassmannian. In this talk we continue the study of Gr_G . Give the Cartan decomposition of Gr_G , explain how to construct the Schubert cells Gr_G^λ for $\lambda \in X_*(T)^+$, explain the proof that they are smooth quasi-projective of dimension $\langle 2\rho, \lambda \rangle$ and describe the associated Schubert variety, cf. [21, Proposition 2.1.5] (see also [13, §§2.3 & Thm 2.15] for the case $\mathrm{GL}_n(\mathbb{C})$). Give a description of the connected components of Gr_G , cf. [21, Theorem 1.3.11] and [13, Prop 2.3.8], and explain the distinction between odd/even components as in the discussion following [1, Prop. 3.2]. Finally, sketch the construction of the semi-infinite orbits S_μ ($\mu \in X_*(T)$) and the description of their boundary as in [1, Prop. 3.4].

Talk 4 (18.11): Background on perverse sheaves. The aim of this talk is to give an overview of the classical theory of perverse sheaves. It is independent of the previous talks. Define local systems, (Whitney) stratifications and constructible sheaves on a complex variety X . Then define the derived category of constructible sheaves with respect to a given stratification and recall the usual six functors on it (this is all covered in [13, §§3.1-3.2]). Next, recall the notion of a t -structure (and its heart) and define the perverse t -structure and perverse sheaves, and explain the formal properties of the category of perverse sheaves: it is abelian, objects have finite length, and describe its simple objects (see [13, §§3.4-3.5]). The references [16, 12, 19] and [8, §§8.1] can also provide useful additional details or motivation.

Talk 5 (25.11): Semisimplicity of the Satake category. This talk introduces the Satake category and proves that it is semisimple abelian. First, recall the notion of constructible sheaves with respect to a Whitney stratification for an ind-variety and define the Satake category as the category of perverse sheaves on Gr_G with respect to the stratification given by the Cartan decomposition, cf. [13, §§3.3] and [1, §§4.1]. The main goal of the talk is to then prove [1, Thm 4.2]. Try to give the main ideas and ingredients. Finally, sketch the notion of equivariant perverse sheaves and show that the Satake category (as defined above) is equivalent to the category of L^+G -equivariant perverse sheaves on Gr_G , cf. [1, Cor 4.8 & Appendix].

Talk 6 (02.12): Tannakian reconstruction. This talk is purely formal and independent of the rest of the seminar. Its aim is to introduce the Tannakian formalism, which formally classifies when a symmetric monoidal category is equivalent to the representation category of an algebraic group. First, recall the language of Hopf algebras and the realisation of the representation category of an algebraic group G as the category of comodules over its algebra of functions, cf. [10, §§3.c & 4.a]. The first main observation is next that the group G may be reconstructed from the forgetful functor from $\mathrm{Rep}(G)$ to vector spaces, see [3, Prop 2.8]. The main goal is then to explain the statement and proof of [1, Theorem 2.7], see also [3, Thm 2.11].

Talk 7 (09.12): Weight functors. When we will later reconstruct the dual group from the Satake category, we will have as a step to construct the dual torus. This is done by using the so-called weight functors, which this talk introduces. First, explain that the cohomology of the perverse sheaves in the Satake category is controlled by the semi-infinite orbits of Talk 3 and use this to define the weight functors F_μ with $\mu \in X_*(T)$, see [1, Prop 5.6] and the discussion following its proof. Then show that the total cohomology is obtained from the weight functors and is fully faithful, cf. [1, Thm 5.9]. This is crucial as this will be our fiber functor later. Finally, describe the image of the $\mathbf{IC}$ sheaves under the weight functors, cf. [1, Prop 5.13]. Time permitting, you may discuss as well the independence of this construction from the choice of maximal torus as in [1, §§5.5].

Talk 8 (16.12): Convolution product I. This talk introduces the convolution product and shows that the convolution of perverse sheaves is still perverse (this was referred to as a ‘miraculous theorem’ by Beilinson-Drinfeld). First explain carefully the definition of convolution and show that it is associative, cf. [1, §§6.2, 6.4 & A.3]. Then explain the proof that convolution preserves perversity, see [1, Cor 5.4, §§6.1 & 6.3].

Talk 9 (06.01): Convolution product II. This talk aims to show the commutativity of the convolution product. For this, first define the Beilinson-Drinfeld Grassmannian, motivating it from the moduli description of Gr_G and the Beauville-Laszlo theorem, cf. [1, §§7.1-7.3] and [14, §§3.2]. Then prove the isomorphism [1, eq. (7.8)], based on the technical lemmas from section 7.5 of *loc. cit.*, which identifies the convolution product with a construction on the Beilinson-Drinfeld affine Grassmannian called fusion product. Finally, deduce the commutativity of convolution as in [1, §§7.6].

Talk 10 (13.01): The fiber functor on Sat_G . After the last two talks, the Satake category is now known to be symmetric monoidal. This talk finishes the proof that this category is Tannakian by establishing the final required properties of the fiber functor: show that the fiber functor is monoidal in a way compatible with the commutativity constraint, cf. [1, Prop 8.1 & §§8.2], which relies on a similar statement for the weight functors, cf. [1, Prop 8.3].

Talk 11 (20.01): The proof of the main theorem. After everything we have done, the Satake category is equivalent to the category of representations of some k -group scheme $\tilde{G}$. It remains to show that $\tilde{G} \cong G_k^\vee$. First establish the criteria for algebraicity of a k -group scheme (i.e. that it has finite type) and connectedness from its category of representations in [1, Prop 2.11]. Then show that $\tilde{G}$ is a connected reductive algebraic group and identify its maximal torus with $T_k^\vee$, cf. [1, §§9.1]. Then finish by proving that the root data of $\tilde{G}$ and $G_k^\vee$ coincide, see [1, Thm 9.8].

Talk 12 (27.01): Relation to the Langlands program. The final talk should aim to give an overview of how the geometric Satake equivalence relates to the wider Langlands program. We give below an idea for what this talk could cover but there are various options the speaker could go for, depending on one's taste and bravery, and they are encouraged to reach out to the organiser about what ideas they might have for this talk.

One could first describe the ℓ -adic analogue of the equivalence and explain how, using the function-sheaf dictionary, one can recover the classical Satake isomorphism from the geometric equivalence, cf. [21, §§5.6] and the references therein. One could then explain how the unramified local Langlands correspondence is an easy corollary of that, see e.g. [4, Corollary 7.6]². Finally, following e.g. [18], you could explain what the geometric Langlands correspondence is and how geometric Satake may be viewed as a local (in the geometric sense) version of it. The enthusiastic speaker could discuss other related recent developments of the local (geometric) Langlands program (e.g. Witt vector affine Grassmannians or geometric Satake over B_{dR}^+ to name just a few), but beware that those would easily deserve a whole seminar on their own.

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²Apologies for this shameless act of self-promotion

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