Proof of Katz's theorem

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$$\underset{\text{SNCD}}{D \xhookrightarrow{i}} X \underset{\text{smooth}}{\xrightarrow{f}} S \underset{\text{smooth}}{\xrightarrow{g}} T \qquad (\text{Sometimes } T \text{ is the spectrum of a f.g. subring of } \mathbb{C}.)$$

$$0 \to f^*\Omega^1_{S/T} \to \Omega^1_{X/T}(\log D) \to \Omega^1_{X/S}(\log D) \to 0.$$

Def The Kodaira–Spencer class is the associated element

$$\rho \in \mathrm{Ext}^1(\Omega^1_{X/S}(\log D), f^*\Omega^1_{S/T})$$
$$\simeq H^1(X, \mathrm{Der}_D(X/S) \otimes_{\mathcal{O}_X} f^*\Omega^1_{S/T}).$$

Const

$$\Omega^1_{X/S}(\log D) \xrightarrow{\rho} f^*\Omega^1_{S/T}[1]$$
$$\Omega^p_{X/S}(\log D) \xrightarrow{\rho} f^*\Omega^1_{S/T} \otimes_{\mathcal{O}_X} \Omega^{p-1}_{X/S}(\log D)[1]$$

$$\left(u : V \to W, \quad \begin{array}{ccc} \wedge^p V & \to & W \otimes \wedge^{p-1} V \\ f_1 \wedge \cdots \wedge f_p & \mapsto & \sum_{i=1}^{p} (-1)^{i-1} u(f_i)\, f_1 \wedge \cdots \wedge \hat{f_i} \wedge \cdots \wedge f_p \end{array} \right)$$

$$\rho : R^q f_* \Omega^p_{X/S}(\log D) \to R^q f_*(f^*\Omega^1_{S/T} \otimes_{\mathcal{O}_X} \Omega^{p-1}_{X/S}(\log D)[1])$$
$$\simeq \Omega^1_{S/T} \otimes_{\mathcal{O}_S} R^{q+1} f_* \Omega^{p-1}_{X/S}(\log D).$$

Const (Talk 4) Recall the Griffiths transversality

$$F^p R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^{p-1} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D).$$

$$\rightsquigarrow \mathrm{gr}^p R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathrm{gr}^{p-1} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D).$$

Prop 1 If the Hodge to de Rham spectral seq

$$E_1^{p,q} = R^q f_* \Omega^p_{X/S}(\log D) \Rightarrow R^{p+q} f_* \Omega^\bullet_{X/S}(\log D)$$

degenerates at $E_1^{p,q}$ (true if char $S = 0$), then $\nabla = \rho$.

Recall (now S has char p)

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rr, bend left, "F_{abs}"] \arrow[dr, "f"'] & X^{(p)} \arrow[r, "\sigma"] \arrow[d, "f^{(p)}"] & X \arrow[d, "f"] \\ & S \arrow[r, "F_{abs}"] & S \end{tikzcd}$$

Thm 2 There is a unique isomorphism of $\mathcal{O}_{X^{(p)}}$-modules

$$C^{-1} : \Omega^i_{X^{(p)}/S}(\log D) \to \mathcal{H}^i(F_* \Omega^\bullet_{X/S}(\log D)) \quad (\text{inverse log Cartier operator})$$

satisfying $C^{-1}(1) = 1$, $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$,

$C^{-1}(d(\sigma^{-1}(x))) =$ the class of $x^{p-1}dx$.

Pf Zariski locally on X, there exists an étale morphism

$$X \xrightarrow{u} S \times \mathbb{A}^n \quad \text{s.t.} \quad D = u^{-1}(\underbrace{z_1 + \cdots + z_r}_{\text{axes}}).$$

$X \xrightarrow{u} S \times \mathbb{A}^n$ s.t. $D = u^{-1}(\underbrace{z_1 + \cdots + z_r}_{\text{axes}})$.

We reduce to the case where $X = S \times \mathbb{A}^n$, $D = z_1 + \cdots + z_r$.
We reduce to the case where $S = \mathrm{Spec}(\mathbb{F}_p)$.

Then both sides are explicit. $\square$

<u>Const</u> $E_2^{p,q} = R^p f_* \mathcal{H}^q \Omega^\bullet_{X/S}(\log D) \longrightarrow R^{p+q} f_* \Omega^\bullet_{X/S}(\log D)$

$$
\begin{aligned}
&\cong R^p f^{(p)}_* F_* \mathcal{H}^q(\Omega^\bullet_{X/S}(\log D)) \\
&\cong R^p f^{(p)}_* \mathcal{H}^q(F_* \Omega^\bullet_{X/S}(\log D)) && \text{since } F \text{ is a homeomorphism} \\
&\cong R^p f^{(p)}_* \Omega^q_{X^{(p)}/S}(\log D^{(p)}) && \text{by the log Cartier isom} \\
&\cong R^p f^{(p)}_* \sigma^* \Omega^q_{X/S}(\log D) \\
&\cong F_{abs}^* R^p f_* \Omega^q_{X/S}(\log D) && \text{assume } F_{abs}: S \to S \text{ flat (e.g. } S \text{ regular)}
\end{aligned}
$$

"conjugate spectral seq"

<u>Prop 3</u> Assume
(i) X proper
(ii) $R^p f_* \Omega^q_{X/S}(\log D)$ locally free sheaf of finite rank $\forall p,q$
(iii) The Hodge to de Rham spectral seq degenerates at $E_1^{p,q}$.

Then the conjugate spectral seq deg at $E_2^{p,q}$.

<u>Pf</u> Reduce to the following cases:

- $S = \mathrm{Spec}(A)$
- A finitely generated $\mathbb{Z}$-alg
  (In particular, A is noetherian)
- noetherian local
- complete noetherian local
- artinian local (A/m^{n+1})

Then we can use the lengths of finitely generated A-modules. $\square$

<u>Recall</u> (M, ∇) flat connection.
(Talk 5) Its <u>p-curvature</u> is $\psi: \mathrm{Der}(S/T) \to \mathrm{End}_{\mathcal{O}_S}(M)$
$\delta \longmapsto \nabla(\delta)^p - \nabla(\delta^p)$

$\rightsquigarrow \psi: M \to F_{abs}^*(\Omega^1_{S/T}) \otimes_{\mathcal{O}_S} M$.

The p-curvature is 0 iff $M \cong F_{abs}^* N$ for some N and ∇ = canonical connection.

<u>Const</u> Assume (i)–(iii) above.

$$F^p_{con} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^p_{con} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D)$$

(stronger than the Griffiths transversality)

$$F^p_{con} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\psi} F_{abs}^* \Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^p_{con} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D).$$

$$\mathrm{gr}^p_{\mathrm{con}} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathrm{gr}^p_{\mathrm{con}} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D)$$

has p-curvature 0.

$$\rightsquigarrow F^p_{\mathrm{con}} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\psi} F^*_{\mathrm{abs}} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^{p+1}_{\mathrm{con}} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D)$$

$$\rightsquigarrow \mathrm{gr}^p_{\mathrm{con}} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D) \xrightarrow{\psi} F^*_{\mathrm{abs}} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathrm{gr}^{p+1}_{\mathrm{con}} R^{p+q} f_* \Omega^\bullet_{X/S}(\log D).$$

Thm 4 $\psi = (-1)^{q+1} F^*_{\mathrm{abs}}(\rho)$

$(\rho : R^q f_* \Omega^p_{X/S}(\log D) \to R^{q+1} f_* \Omega^{p-1}_{X/S}(\log D))$

Now, $D \hookrightarrow X \to S \to T$, $U \subset S$

$\exists U$ open dense affine s.t. $R^p f_* \Omega^q_{X/S}(\log D)|_U$ locally free, finite type $\forall p,q$.

Then the Hodge to de Rham spectral seq deg at E_1.

(Let's assume $U = S$)

Thm 5 Assume (i)–(iii), $S = \mathrm{Spec}(A)$.
Let Σ be a set of infinite numbers of primes.
Assume that for every finite field $\mathbb{F}_q$ of char p and morphism $\mathrm{Spec}(\mathbb{F}_q) \to T$,
the p-curvature of $(R^q f_* \Omega^p_{X/S}(\log D) \otimes \mathbb{F}_q, \nabla)$ is 0.

Then $(R^q f_{\mathbb{C}*} \Omega^p_{X_{\mathbb{C}}/S_{\mathbb{C}}}(\log D_{\mathbb{C}}), \nabla)$ becomes trivial after a finite étale covering of $S_{\mathbb{C}}$. $\quad (*)$

Proof $(*)$
(Talk 7)
$\Leftarrow$ The Hodge filtration on $R^q f_{\mathbb{C}} \Omega^p_{X_{\mathbb{C}}/S_{\mathbb{C}}}(\log D_{\mathbb{C}})$ is horizontal
$\Leftarrow$ The Hodge Filtration on $R^q f \Omega^p_{X/S}(\log D)$ is horizontal
(Prop 1)
$(\Rightarrow)$ For every $\delta \in \mathrm{Der}(S/T)$,

$$\bigoplus_{p+q=n} R^q f_* \Omega^p_{X/S}(\log D) \xrightarrow{\rho(\delta)} \bigoplus_{p+q=n} R^{q+1} f_* \Omega^{p-1}_{X/S}(\log D)$$

vanishes

Thm 4 $\Rightarrow$ $\rho(\delta)$ becomes 0 after base change to $\mathbb{F}_q$.

Say $S = \mathrm{Spec}(A)$.

The matrix $\rho(\delta)$ has coefficients in $\mathfrak{m}$ $\forall$ maximal $\mathfrak{m}$ such that $A/\mathfrak{m}$ has char in Σ.

Noether's normalization theorem implies

$$\bigcap_{\mathrm{char}\, A/\mathfrak{m} \in \Sigma} \mathfrak{m} = 0 \qquad (\exists\, n \in \mathbb{N}^+ \text{ and a finite inj hom } \mathbb{Z}_{(n)}[x_1, \ldots, x_d] \hookrightarrow A_{(n)})$$

Hence $p(S)=0$. $\square$

Hypergeometric equation:

$$z(1-z)\frac{d^2w}{dz^2}+(c-(a+b+1)z)\frac{dw}{dz}-abw=0$$

Sol: ${}_2F_1\left(\begin{matrix}a\ b\\ c\end{matrix};z\right)$, $z^{1-c}{}_2F_1\left(\begin{matrix}a-c+1 \quad b-c+1\\ 2-c\end{matrix};z\right)$ $(z\neq 0)$

$${}_2F_1\left(\begin{matrix}a\ b\\ c\end{matrix};z\right):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad (a)_n:=a(a+1)\cdots(a+n-1)$$

Ex ${}_2F_1\left(\begin{matrix}a\ b\\ b\end{matrix};z\right)=(1-z)^{a}$

$${}_2F_1\left(\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ \frac{3}{2}\end{matrix};z\right)=\frac{3\sin\left(\frac{1}{3}\arcsin(\sqrt{z})\right)}{\sqrt{z}}\qquad {}_2F_1\left(\begin{matrix}-\frac{1}{6}\ \frac{1}{6}\\ \frac{1}{2}\end{matrix};z\right)=\cos\left(\frac{1}{3}\arcsin(\sqrt{z})\right)$$

$${}_2F_1\left(\begin{matrix}1\ 1\\ 2\end{matrix};z\right)=\frac{\ln(1+z)}{z}$$

$${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ \frac{3}{2}\end{matrix};z\right)=\frac{\arcsin(z)}{z}$$

$${}_2F_1\left(\begin{matrix}\frac{47}{8}\ \frac{49}{8}\\ -\frac{41}{8}\end{matrix};z\right)=\frac{(21990232555552z^{11}+\cdots)}{969073995(1-z)^{\frac{135}{8}}}$$

$S:=(\mathbb{A}^1-\{0,1\})\times T$ $\quad\lambda$ coordinate on $\mathbb{A}^1$

Def $E(a,b,c)$ the free $\mathcal{O}_S$-module basis $\{e_0,e_1\}$

$$\nabla(e_0):=e_1\,d\lambda$$

$$\nabla(e_1):=-\frac{c-(a+b+1)\lambda}{\lambda(\lambda-1)}e_1\,d\lambda+\frac{ab}{\lambda(1-\lambda)}e_0\,d\lambda$$

$(E^{\nabla})^*=\{\text{solutions}\}$

Thm 6 TFAE

(i) The hypergeometric equation has a full set of alg sols

(ii) The inverse image $E(a,b,c)$ on $\mathbb{C}(\lambda)$ has p-curvature 0 for almost all primes p.

(iii) $a,b,c\in\mathbb{Q}$, and for almost all primes p, the hypergeometric equation has two solutions in $\mathbb{F}_p(\lambda)$ linearly indep over $\mathbb{F}_p(\lambda^p)$.

Def $A(n;a,b,c):=\mathbb{C}(\lambda)[x,y,\frac{1}{y}]/(y^n-x^a(x-1)^b(x-\lambda)^c)$

finite étale over

$B:=\mathbb{C}(\lambda)[x,\frac{1}{x(x-1)(x-\lambda)}]$

with basis $1,\frac{1}{y},\ldots,\frac{1}{y^{n-1}}$

$X(n;a,b,c):=\operatorname{Spec}(A(n;a,b,c))$

$$H^1_{dR}(X(n;a,b,c)/\mathbb{C}(\lambda)) = \operatorname{coker}\left(A(n;a,b,c) \xrightarrow{d} A(n;a,b,c)\,dx \right)$$

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \end{array}$$

$$P(\chi(-\ell))H^1_{dR}(X(n;a,b,c)/\mathbb{C}(\lambda)) = \operatorname{coker}\left(y^{-\ell} B \xrightarrow{d} y^{-\ell} B\,dx \right)$$

Prop 7 $n, a, b, c \in \mathbb{N}^+$, n does not divide a, b, c, or $a+b+c$.

Then for any $\ell \in \mathbb{N}^+$ coprime to n,

$$E\left(\frac{\ell c}{n}, \frac{\ell a+\ell b+\ell c}{n} - 1, \frac{\ell a+\ell c}{n}\right) \Big| \mathbb{C}(\lambda)$$

$$\simeq P(\chi(-\ell))H^1_{dR}(X(n;a,b,c)/\mathbb{C}(\lambda))$$