Forschungsseminar WS 23/24

Smooth mod p representations of p-adic reductive groups[1]

The study of smooth mod p representations of a p-adic reductive group was initiated by Barthel–Livné in 1994. For a finite extension $F/\mathbb{Q}_p$ they proved that the smooth irreducible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_p$ that admit a central character fall into four classes: (i) characters, (ii) twists of Steinberg representations, (iii) irreducible principal series representations, and (iv) supersingular representations. The latter remain mysterious; by work of Breuil they are completely known for $\mathrm{GL}_2(\mathbb{Q}_p)$. But already for $\mathrm{GL}_2(F)$, $F \neq \mathbb{Q}_p$, the classification of supersingular representations turns out to be vastly more complicated. In his landmark paper [Her11] Herzig classified the irreducible admissible representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{F}}_p$ in terms of parabolic induction and supersingular representations. While supersingular representations are defined via the mod-p Satake transform, the classification shows that they coincide with the supercuspidal representations, *i.e.*, those that are not a subquotient of a parabolically induced representation. This classification was recently generalized to arbitrary connected reductive groups by Abe–Henniart–Herzig–Vignéras [AHHV16].

In this seminar we concentrate on the group $\mathrm{GL}_n(F)$ and follow mostly Herzig's paper [Her11]. Herzig's lecture notes [Her12] are a useful complement to his paper.

1) **Discussion and distribution of the talks.**
2) **Smooth mod p representations and weights.** Introduce basic notions and results from the mod p representation theory of p-adic groups following [Her12], §2 and §3: induced representations, Frobenius reciprocity, the p-group lemma (Lem. 7), weights etc. Cover [Her11] §2.1. More results on highest weights and modular representations of finite groups can be found in [Hum05].
3) **The mod p Satake transform.** Cover [Her11] §2.2–§2.3. Give as many details as possible from [Her10, §2.1] on the construction of the mod p Satake transform and its basic properties.
4) **Various lemmas.** Discuss [Her11] §2.4. Input from Bruhat–Tits theory, e.g. in [Her11, Lem. 2.16], can be stated without proof. In the proof of [Her11, Lem. 2.20] state the relevant properties of generalized Tits systems, without giving the full definition. Present instead the example for GL_n at the end of [Iwa65, §2]. If time is runnig out, one may skip [Her11, Cor. 2.19].
5) **Compatibilities between Hecke actions.** Cover [Her11, §2.5]. In this talk various subalgebras of Hecke algebras are introduced and identified [Her11, Lem.'s 2.21 and 2.22]. The goal of this talk is to prove [Her11, Cor. 2.25] which constructs isomorphisms between certain compactly induced representations.
6) **Comparison of compact induction with parabolic induction.** Prove [Her11, Thm. 3.1 and Cor. 3.6].
7) **Hecke eigenvalues and supersingularity.** Cover [Her11] §§4 and 5. In particular, define supersingular representations [Her11, Def. 4.7] and prove the different characterizations. Sketch the proof of [Her11, Prop. 5.1].
8) **Change of weight.** Cover [Her11] §§6.1 and 6.2 until Prop. 6.7. If time is running out, one may skip [Her11, Cor. 6.5].
9) **Generalized Steinberg representations.** Finish [Her11, §6], in particular present Cor. 6.10 and Ex. 6.14. (You may skip [Her11, Prop. 6.13].) Introduce the generalized Steinberg representations Sp_P [Her11, §7] and prove that they are irreducible and admissible [Her11, Thm. 7.2] (also consult [GK14, §3]). Deduce [Her11, Cor. 7.3]. Show that Sp_P contains a unique weight V_P with multiplicity one and determine the Hecke eigenvalues of V_P [Her11, Prop. 7.4]. Finally determine the Jordan-Hölder factors of $\mathrm{Ind}_{\overline{P}}^{G} \mathrm{Sp}_Q$ [Her11, Prop. 7.6].

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10) Irreducibility of parabolic inductions. Prove the criterion for the irreducibility of parabolic inductions [Her11, Thm. 8.1] and its generalization [Her11, Thm 8.6]. Show that general parabolically induced representations are of finite length and determine the subquotients [Her11, Thms. 8.5 and 8.7]. If time permits, prove the general [Her11, Thm. 8.8].

11) The right adjoint of parabolic induction. Define the ordinary parts functor Ord_P [Eme10, 3.1.9. Def.] and sketch the proof of the fact [Eme10, Thm. 4.4.6] that Ord_P is right adjoint to $\mathrm{Ind}_{\overline{P}}^{G}$. (In our setting, the ring A is a field.) Finally prove [Her11, Prop. 9.1].

12) Classification of irreducible representations. Finish [Her11] §9.1 and discuss §9.2 until Lem. 9.16. In particular, give the classification of irreducible admissible $\mathrm{GL}_n(F)$-representations [Her11, Thm. 9.8] and prove [Her11, Cor.'s 9.10, 9.11, 9.13].

13) * **The submodule structure of parabolically induced representations.** Prove [Her11, Thm. 9.17]; references to Bruhat–Tits theory can be treated very lightly. Then discuss [Her11, §10].

14) * **Ausblick: Classification for general reductive groups.** The aim of this talk is to give an overview of the classification of irreducible admissible representations of a general connected reductive group. Cover the introduction in [AHHV16]. In particular, explain how the description of the irreducible admissible representations in terms of triples (P, σ, Q) relate to the description in talk **12)**.

References

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