One ring to rule them all

by Julian Reichardt

Three rings for the Elven-kings under the sky,
$\mathrm{B}_{\mathrm{cris}}$, B_{st}, B_{dR}
Seven for the Dwarf-lords in their halls of stone,
$\tilde{\mathrm{A}}$, $\mathrm{E}_{\mathbb{Q}_p}$, $\mathrm{A}_{\mathbb{Q}_p}$, $\mathrm{B}_{\mathbb{Q}_p}$, E, A, B,
Nine for the mortal Men doomed to die,
$\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{F}_p$, $\overline{\mathbb{Q}}_p$, $\mathbb{C}_p$, $\mathcal{O}_{\mathbb{C}_p}$, $\mathbb{Q}_p^{\mathrm{unr}}$, B_{HT},
One ring to rule them all,
$\mathbb{A}_{\mathrm{inf}}$.
P. Colmez [Col19]

Introduction

Classical Hodge theory states that for a smooth projective complex variety X, the singular cohomology of its analytification X^{an} can be computed in terms of its Hodge cohomology, after base changing the singular cohomology to the complex numbers. This generalises to smooth proper K-schemes, for $K/\mathbb{Q}_p$ finite, where the the Hodge–Tate decomposition shows that after base changing to a completed algebraic closure $C = \widehat{\overline{K}}$, the étale cohomology of $X_{\overline{K}}$ can be identified with the Hodge cohomology of X base changed to C. This can even be done equivariantly for the action of the absolute Galois group G_K of K, by taking Tate twists into account for the Hodge cohomology.

The comparison theorems of p-adic Hodge theory aim at establishing analogues results for the various other cohomology theories associated to X and related geometric objects. However, it typically no longer suffices to just base change the étale cohomology of X to C. Instead, one introduces various *period rings* allowing for these comparison results to be hold. Two of the central cases are the *de Rham period ring* B_{dR}, for the algebraic de Rham cohomology of X, and the *crystalline period ring* $\mathrm{B}_{\mathrm{cris}}$, for the crystalline cohomology of $\mathcal{X}_0$, the reduction of a smooth proper integral model $\mathcal{X}$ of X (in the case where X has good reduction). The underlying integral period rings $\mathrm{B}_{\mathrm{dR}}^+$ and $\mathrm{B}_{\mathrm{cris}}^+$ can both be constructed from the same source: *Fontaine's first period ring* $\mathbb{A}_{\mathrm{inf}}$.

Plan:

- define Fontaine's first period ring $\mathbb{A}_{\mathrm{inf}}$
- compare $\mathbb{A}_{\mathrm{inf}}$ with a ring of formal power series $\mathcal{O}_F[\![z]\!]$
- define and study the integral de Rham period ring $\mathrm{B}_{\mathrm{dR}}^+$
- define and study the integral crystalline period ring $\mathrm{B}_{\mathrm{cris}}^+$

Notation: Let p be a fixed prime. Let $E/\mathbb{Q}_p$ denote a finite field extension with ring of integers $\mathcal{O}_E$. Fix a uniformiser $\pi \in \mathfrak{m}_E$ so that the residue field is $\mathcal{O}_E/(\pi) = \mathbb{F}_q$ with $q = p^f$. Let C be a complete nonarchimedean algebraically closed extension of E, with ring of integers $\mathcal{O}_C = \{x \in C \,|\, |x| \leq 1\} \subseteq C$, and maximal ideal $\mathfrak{m}_C = \{x \in \mathcal{O}_C \,|\, |x| < 1\} \leq \mathcal{O}_C$.

1 Fontaine's First Period Ring $\mathbb{A}_{\mathbf{inf}}$

1.1 Defining $\mathbb{A}_{\mathbf{inf}}$

We have fixed a complete nonarchimedean algebraically closed extension C of E. We denote by $O_F = O_C^\flat$ the tilt of its ring of integers and by $F = \mathrm{Frac}(O_C^\flat)$ the fraction field of its tilt. Last talk, we saw that O_C is a perfectoid O_E-algebra and that $C^\flat$ is an algebraically closed extension of $O_E^\flat = \mathbb{F}_q$. Relative our choice of C, we make the following definition.

Definition 1.1. We define **Fontaine's first period ring** by

$$\mathbb{A}_{\mathbf{inf}} := W_{O_E}(O_F)\,. \qquad \lrcorner$$

Remark 1.2. Instead of fixing C as a complete nonarchimedean algebraically closed extension of E, we could have fixed F as a complete nonarchimedean algebraically closed extension of $\mathbb{F}_q$. In this case, we need to choose C as a suitable *untilt* of F, which will be the topic of the next talk. ⌟

Let us briefly explain the notation for $\mathbb{A}_{\mathrm{inf}}$.

Definition 1.3. Let R be a π-complete O_E-algebra. A surjection $D \to R$ of O_E-algebras with kernel I so that D is (I, π)-adically complete is called a **π-adic pro-infinitesimal thickening of R**. ⌟

Proposition 1.4. *Let $R \in \{O_C, O_C/(\pi)\}$. Then $\mathbb{A}_{\mathrm{inf}}$ is the universal π-adic pro-infinitesimal thickening of R, i.e. for each π-adic pro-infinitesimal thickening $D \to R$, there exists a unique morphism $\mathbb{A}_{\mathrm{inf}} \to D$ over R.*

Proof. To prove the proposition, we recall that there is an adjunction

$$\mathrm{Hom}_{\pi\text{-complete } O_E\text{-algebras}}(W_{O_E}(-), -) \cong \mathrm{Hom}_{\text{perfect } \mathbb{F}_q\text{-algebras}}(-, (-)^\flat)$$

with tilting $(-)^\flat$ as right-adjoint and taking ramified Witt vectors $W_{O_E}(-)$ as left-adjoint.

We first check, that $\mathbb{A}_{\mathrm{inf}}$ is a π-adic pro-infinitesimal thickening of R. While proving, that O_C is a perfectoid O_E-algebra, it was established that the counit of the adjunction yields a surjection $\mathbb{A}_{\mathrm{inf}} \to O_C$ with kernel I generated by $\xi = \pi - [\pi^\flat]$. The (I, π)-adic completeness of $\mathbb{A}_{\mathrm{inf}}$ follows from $\mathbb{A}_{\mathrm{inf}}/(I, \pi) \cong O_C/(\pi)$ and the π-adic completeness of O_C. If we consider the induced surjection $\mathbb{A}_{\mathrm{inf}} \to O_C/(\pi)$ with kernel J, we have $(J, \pi) = (I, \pi)$. Thus, $\mathbb{A}_{\mathrm{inf}} \to R$ is a π-adic pro-infinitesimal thickening of R.

Now let $D \to R$ be a π-adic pro-infinitesimal thickening of R. We then compute

$$R^\flat \cong (R/(\pi))^\flat \cong (D/(I, \pi))^\flat \cong D^\flat\,.$$

Thus, there is a unique morphism $\mathbb{A}_{\mathrm{inf}} \to D$ inducing the isomorphism $D^\flat \cong R^\flat$ along the bijections

$$\begin{aligned}\mathrm{Hom}_{\pi\text{-complete } O_E\text{-algebras}}(\mathbb{A}_{\mathrm{inf}}, D) &= \mathrm{Hom}_{\pi\text{-complete } O_E\text{-algebras}}(W_{O_E}(O_C^\flat), D)\\ &\cong \mathrm{Hom}_{\text{perfect } \mathbb{F}_q\text{-algebras}}(O_C^\flat, D^\flat)\\ &\cong \mathrm{Hom}_{\text{perfect } \mathbb{F}_q\text{-algebras}}((O_C/(\pi))^\flat, D^\flat)\,.\end{aligned}$$

In particular, the isomorphism $R^\flat \cong D^\flat$ lifts to a unique morphism $\mathbb{A}_{\mathrm{inf}} \to D$, as needed. □

1.2 Interlude: The Ring $O_F[\![z]\!]$

Since O_F is a perfect $\mathbb{F}_q$-algebra, we know that every element of $\mathbb{A}_{\mathrm{inf}} = W_{O_E}(O_F)$ has unique expression as a power series in π with coefficients in Teichmüller lifts of elements O_F. Although, the ring structure is vastly different from the ring of formal power series over O_F, we will see that $\mathbb{A}_{\mathrm{inf}}$ and $O_F[\![z]\!]$ share a variety of properties.

But let us first review some properties of the ring $O_F[\![z]\!]$.

Definition 1.5. Let

$$\boldsymbol{O_F}[\![z]\!] \coloneqq \left\{ \sum_{n=0}^{\infty} a_n z^n \;\middle|\; a_n \in O_F \right\}$$

be the **ring of formal power series over $\boldsymbol{O_F}$**. ⌟

As O_F is not discretely valued, O_F is non-noetherian and its maximal ideal satisfies $\mathfrak{m}_F^2 = \mathfrak{m}_F$. Using this, one can show that $O_F[\![z]\!]$ has infinite Krull dimension.

Theorem 1.6 ([Arn73]). *The ring $O_F[\![z]\!]$ has infinite Krull dimension.* □

Nonetheless, one can make the prime spectrum of $O_F[\![z]\!]$ more explicit. Of course, one has the zero ideal (0), as O_F is an integral domain, and one has the unique maximal ideal $(\mathfrak{m}_F, z)$, as $O_F[\![z]\!]$ is a formal power series ring over the complete local ring $(O_F, \mathfrak{m}_F)$. Moreover, for each $a \in \mathfrak{m}_F$ and $f \in O_F[\![z]\!]$, the series $f(a)$ converges to a well-defined element of O_F. Thus, one obtains the evaluation morphisms

$$\mathrm{ev}_a \colon O_F[\![z]\!] \longrightarrow O_F, \qquad f(z) \longmapsto f(a)$$

for each $a \in \mathfrak{m}_F$. We claim that their kernels are the prime ideals $(z - a)$ and that there are no other prime ideals not contained in the extension of the maximal ideal $\mathfrak{m}_F[\![z]\!]$. Thus, $O_F[\![z]\!]$ being of infinite Krull dimension comes down to the prime spectrum of the localisation $O_F[\![z]\!]_{\mathfrak{m}_F[\![z]\!]}$ being of infinite Krull dimension.

Proposition 1.7. *The spectrum of $O_F[\![z]\!]$ is given by*

$$\mathrm{Spec}\,(O_F[\![z]\!]) = \{(0), (\mathfrak{m}_F, z)\} \cup \{(z-a) \mid a \in \mathfrak{m}_F\} \cup \mathrm{Spec}\left(O_F[\![z]\!]_{\mathfrak{m}_F[\![z]\!]}\right) .$$

Proof. We use Weierstraß theory for formal power series rings over complete local rings to prove the claimed decomposition. First, the Weierstraß division theorem implies, that if $a \in \mathfrak{m}_F$ and $h \in O_F[\![z]\!]$, then there exists $q \in O_F[\![z]\!]$ and $r \in O_F$ so that $h = q(z-a) + r$. Second, the Weierstraß preparation theorem implies, that if $f = \sum_{n=0}^{\infty} a_n z^n \notin \mathfrak{m}_F[\![z]\!]$, then $f = ug$ with u a unit and g a distinguished polynomial of degree $d = \min\left\{n \mid a_n \in O_F^\times\right\}$, i.e. a monic polynomial of degree $d = \min\left\{n \mid a_n \in O_F^\times\right\}$ whose remaining coefficients are in $\mathfrak{m}_F$.

Observe that the above ideals are all prime. This is clear for the zero ideal (0), as $O_F[\![z]\!]$ is an integral domain, and also for $(\mathfrak{m}_F, z)$, as the quotient $O_F[\![z]\!]/(\mathfrak{m}_F, z)$ can be identified with the residue field of O_F. For the ideals $(z-a)$ with $a \in \mathfrak{m}_F$, we note that these are the kernels of the valuation morphisms $\mathrm{ev}_a \colon O_F[\![z]\!] \to O_F$, which can be seen using the Weierstraß division theorem. In particular, $(z-a)$ is prime for all $a \in \mathfrak{m}_F$. Finally, the ideals in $\mathrm{Spec}\left(O_F[\![z]\!]_{\mathfrak{m}_F[\![z]\!]}\right)$ are prime by definition.

Now let $\mathfrak{p}$ be a prime ideal of $O_F[\![z]\!]$. We assume that $\mathfrak{p} \neq (0)$. We claim that if, furthermore, $\mathfrak{p} \not\subseteq \mathfrak{m}_F[\![z]\!]$, then either there exists $a \in \mathfrak{m}_F$ such that $\mathfrak{p} = (z-a)$, or $\mathfrak{p} = (\mathfrak{m}_F, z)$.

By assumption, there exists $f = \sum_{n=0}^{\infty} a_n z^n \in \mathfrak{p}$ such that $f \notin \mathfrak{m}_F[\![z]\!]$. Thus, by Weierstraß' preparation theorem, there exists a distinguished polynomial g of degree $d = \min\left\{n \mid a_n \in O_F^\times\right\}$ such that $f = ug$ for some unit u. As F is algebraically closed, g decomposes into linear factors and since g is distinguished all its root are elements in $\mathfrak{m}_F$. Since $\mathfrak{p}$ is prime, $x - a \in \mathfrak{p}$ for some $a \in \mathfrak{m}_F$. If $\mathfrak{p} = (z - a)$, we are done. So suppose that $(x - a) \subsetneq \mathfrak{p}$. For $h \in \mathfrak{p}$ not a multiple of $z - a$, Weierstraß division yields $h = q(z - a) + r$ with $r \in \mathcal{O}_F$. Note that $r = f - q(z - a) \in \mathfrak{p}$. Now r cannot be a unit, as $\mathfrak{p}$ is a prime ideal. Thus, $r \in \mathfrak{m}_F$. As F is algebraically closed, $r^{1/m} \in \mathfrak{m}_F$ exists for all $m \geq 1$. Moreover, $r^{1/m} \in \mathfrak{p}$ for all $m \geq 1$ since $\mathfrak{p}$ is prime. Now let $b \in \mathfrak{m}_F$ be arbitrary. As $|b| \leq 1$, there exists m such that $|b| \leq |r^{1/m}|$. In particular, $|b/r^{1/m}| \leq 1$ and $b = (b/r^{1/m})r^{1/m}$ shows that $b \in \mathfrak{p}$. Thus, $(z - a, \mathfrak{m}_F) = (z, \mathfrak{m}_F) \subseteq \mathfrak{p}$, which shows that $\mathfrak{p} = (\mathfrak{m}_F, z)$ as $(\mathfrak{m}_F, z)$ is maximal. □

Finally, we given a geometric interpretation of the ring $\mathcal{O}_F[\![z]\!]$. For this, let

$$\mathbb{D}_F^\circ = \{x \in F \mid |x| < 1\}$$

be the open rigid analytic unit disk. Fix a pseudo-uniformiser $t \in \mathfrak{m}_F$. The sets

$$\mathbb{D}_F^\circ\left(t^{-1/m}\right) = \left\{x \in F \;\middle|\; |x| < |t|^{1/m}\right\}$$

form an open admissible cover of $\mathbb{D}_F^\circ$, thus

$$\begin{aligned}
\mathcal{O}(\mathbb{D}_F^\circ) &= \varprojlim_m \mathcal{O}\left(\mathbb{D}_F^\circ\left(t^{-1/m}\right)\right) \\
&= \varprojlim_m \left\{\sum_{n=0}^\infty a_n z^n \;\middle|\; |a_n||t|^{n/m} \to 0\right\} \\
&= \left\{\sum_{n=0}^\infty a_n z^n \;\middle|\; |a_n|\rho^n \to 0 \text{ for all } \rho < 1\right\}.
\end{aligned}$$

Note that if $f \in \mathcal{O}_F[\![z]\!]$, then $f \in \mathcal{O}(\mathbb{D}_F^\circ)$. Conversely, if $f \in \mathcal{O}(\mathbb{D}_F^\circ)$ is uniformly bounded by 1, then $f \in \mathcal{O}_F[\![z]\!]$. Thus, $\mathcal{O}_F[\![z]\!]$ is the ring of bounded rigid analytic functions on the open unit disk.

1.3 Basic Properties of $\mathbb{A}_{\mathrm{inf}}$

We now go over some basic properties of $\mathbb{A}_{\mathrm{inf}}$, mirroring our discussion of $\mathcal{O}_F[\![z]\!]$.

For starters, the ring $\mathbb{A}_{\mathrm{inf}}$ is a local domain, by virtue of being constructed as a ring of (ramified) Witt vectors. Moreover, $\mathbb{A}_{\mathrm{inf}}$ is of infinite Krull dimension as well.

Theorem 1.8 ([LL21]). *The ring $\mathbb{A}_{\mathrm{inf}}$ has infinite Krull dimension.* □

There is an analogue of the function-theoretic interpretation we gave for $\mathcal{O}_F[\![z]\!]$. However, this uses the adic Fargues–Fontaine curve, which we will no introduce. Instead, we consider the following weak analogue.

Definition 1.9. We define

$$\boldsymbol{|Y|_{[0,\infty)}} := \left\{I \subsetneq \mathbb{A}_{\mathrm{inf}} \mid I \text{ is generated by a distinguished element}\right\}$$

and set

$$\boldsymbol{|Y|} := |Y|_{[0,\infty)} \setminus \{(\pi)\}.$$ ⌟

Remark 1.10. The set $|Y|$ will also be of interest when studying the possible untilts of F (cf. Remark 1.2). ⌟

The distinguished elements appearing in Definition 1.9 admit a more explicit description.

Lemma 1.11. *A distinguished element $d \in \mathbb{A}_{\mathrm{inf}}$ generates a proper ideal $I = (d)$ if any only if $d = [a] + \pi u$ with $a \in \mathfrak{m}_F$ and $u \in \mathbb{A}_{\mathrm{inf}}{}^{\times}$. Thus,*

$$|Y|_{[0,\infty)} := \left\{(u\pi - [a]) \,|\, u \in \mathbb{A}_{\mathrm{inf}}^{\times}, a \in \mathfrak{m}_F\right\} .$$

Proof. Recall that we can write $d \in \mathbb{A}_{\mathrm{inf}} = W_{O_E}(O_F)$ as $d = \sum_{n=0}^{\infty}[d_n]\pi^n$. It is then known, that d is distinguished if and only if $d_1 \in O_F^{\times}$. Moreover, d is a unit, hence the ideal $I = (d)$ is the unit ideal, if and only if $d_0 \in O_F^{\times}$. So if $I = (d)$ is a proper ideal and d is required to be distinguished, then $d_0 \in \mathfrak{m}_F$ and $d_1 \in O_F^{\times}$. Thus, we may write

$$d = \sum_{n=0}^{\infty}[d_i]\pi^n = [d_0] + \pi\sum_{n=0}^{\infty}[d_{n+1}]\pi^n .$$

Now let $a = -d_0$ and $u = \sum_{n=0}^{\infty}[u_n]\pi^n$ for $u_n = d_{n+1}$. As $d_1 \in O_F^{\times}$ by assumption, $u \in \mathbb{A}_{\mathrm{inf}}$. Moreover, one checks that $[-a] = -[a]$. Therefore, we can write $d = u\pi - [a]$, as claimed.

Conversely, any such element is distinguished and not a unit, hence generates a proper ideal $I \in |Y|_{[0,\infty)}$. □

We close this section by recalling, that for every choice $\pi^{\flat} = (\pi, \pi^{1/q}, \dots)$, let $\xi = \pi - [\pi^{\flat}]$. Then there is an isomorphism

$$\mathbb{A}_{\mathrm{inf}}/(\xi) = \mathbb{A}_{\mathrm{inf}}/(\pi - [\pi^{\flat}]) \cong O_C .$$

This realises O_C as a perfectoid O_E-algebra associated to the perfect prism $(\mathbb{A}_{\mathrm{inf}}, (\xi))$.

2 The De Rham and Crystalline Period Rings

2.1 The Integral De Rham Period Ring $\mathbf{B}_{\mathbf{dR}}^{+}$

We now introduce the de Rham period ring, constructed from the perfect prism $(\mathbb{A}_{\mathrm{inf}}, (\xi))$.

Definition 2.1. We define the **integral de Rham period ring** as

$$\mathbf{B}_{\mathbf{dR}}^{+} := \mathbb{A}_{\mathrm{inf}}[\pi^{-1}]_{\xi}^{\wedge} .$$

The field $\mathrm{B}_{\mathrm{dR}} = \mathrm{Frac}(\mathrm{B}_{\mathrm{dR}}^{+})$ is called the **de Rham period ring**, or **Fontaine's field of *p*-adic periods for *C***. ⌟

Since $\mathrm{B}_{\mathrm{dR}}^{+}$ is the ξ-adic completion of $\mathbb{A}_{\mathrm{inf}}[\pi^{-1}]$, it is a ξ-adically complete ring so that

$$\mathrm{B}_{\mathrm{dR}}^{+}/(\xi) \cong \mathbb{A}_{\mathrm{inf}}[\pi^{-1}]/(\xi) \cong C .$$

On the other hand, we can endow each $\mathbb{A}_{\mathrm{inf}}[\pi^{-1}]/(\xi^n)$ with a unique topology such that $\mathbb{A}_{\mathrm{inf}}/(\xi^n)$ is open in the π-adic topology. The resulting inverse limit topology on $\mathrm{B}_{\mathrm{dR}}^{+}$ is called the **canonical topology**.

Proposition 2.2. *The natural morphism $\mathbb{A}_{\mathrm{inf}} \to \mathrm{B}_{\mathrm{dR}}^{+}$ is injective.*

Proof. As $\mathbb{A}_{\mathrm{inf}}$ is $(\pi, [\pi^{\flat}])$-complete and $(\xi) \subseteq (\pi, [\pi^{\flat}])$ is finitely generated, $\mathbb{A}_{\mathrm{inf}}$ is ξ-adically complete. Thus, it suffices to show that the natural localisation maps $\mathbb{A}_{\mathrm{inf}}/(\xi^n) \to \mathbb{A}_{\mathrm{inf}}[\pi^{-1}]/(\xi^n)$ are injective. Indeed, the map $\mathbb{A}_{\mathrm{inf}} \to \mathrm{B}_{\mathrm{dR}}^{+}$ arises as the projective limit of the maps $\mathbb{A}_{\mathrm{inf}}/(\xi^n) \to \mathbb{A}_{\mathrm{inf}}[\pi^{-1}]/(\xi^n)$ and $\varprojlim$ is left-exact.

Since $\xi \in \mathbb{A}_{\mathrm{inf}}$ is not a zero divisor ($\mathbb{A}_{\mathrm{inf}}$ being an integral domain) and O_C is π-torsionfree, using induction and the short exact sequences

$$0 \longrightarrow (\xi^{n-1})/(\xi^n) \longrightarrow \mathbb{A}_{\mathrm{inf}}/(\xi^n) \longrightarrow \mathbb{A}_{\mathrm{inf}}/(\xi^{n-1}) \longrightarrow 0$$

we see that $\mathbb{A}_{\inf}/(\xi^n)$ is π-torsionfree for all $n \geq 1$. In particular, $\mathbb{A}_{\inf}/(\xi^n) \to \mathbb{A}_{\inf}[\pi^{-1}]/(\xi^n)$ is injective. $\square$

Proposition 2.3. *The local ring* $\mathrm{B}_{\mathrm{dR}}^{+}$ *is a discrete valuation ring.*

Proof. As $\mathrm{B}_{\mathrm{dR}}^{+}$ is the ξ-adic completion of $\mathbb{A}_{\inf}[\pi^{-1}]$ and $\mathbb{A}_{\inf}[\pi^{-1}]/(\xi) \cong C$ is a field, hence, in particular noetherian, one can show that $\mathrm{B}_{\mathrm{dR}}^{+}$ is noetherian. Moreover, $\mathrm{B}_{\mathrm{dR}}^{+}$ is a local domain with maximal ideal generated by ξ. Thus, by Krull Hauptidealsatz, $\mathrm{B}_{\mathrm{dR}}^{+}$ is of Krull dimension 1. Thus, $\mathrm{B}_{\mathrm{dR}}^{+}$ is a discrete valuation ring. $\square$

Remark 2.4. As $\mathrm{B}_{\mathrm{dR}}^{+}$ is a complete discrete valuation ring whose residue field C is of characteristic 0, Cohen's structure theorem implies that $\mathrm{B}_{\mathrm{dR}}^{+}$ is abstractly isomorphic to $C[\![z]\!]$. Moreover, using that $\mathrm{B}_{\mathrm{dR}}^{+}$ is a discrete valuation ring, one can deduce that the localisation $(\mathbb{A}_{\inf})_{(\xi)}$ is a discrete valuation ring as well. As this is not needed anywhere, we omit the proof. ⌟

2.2 The Integral Crystalline Period Ring $\mathrm{B}_{\mathrm{cris}}^{+}$

We now introduce the crystalline period ring, constructed from the perfect prism $(\mathbb{A}_{\inf}, (\xi))$ as well. However, for this we restrict to the case $E = \mathbb{Q}_p$ and $\pi = p$.

We recall the notions associated to divided power structures, which are foundational to the theory of crystalline cohomology.

Definition 2.5. Let A be a ring and $I \leq A$ an ideal. A **divided power structure on** $\boldsymbol{I}$, or **PD-structure**, is a collection of maps $(\gamma_n \colon I \to A)_{n\geq 0}$ such that

(i) $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_n(x) \in I$ for all $x \in I$ and $n \geq 2$,

(ii) $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$ for all $x, y \in I$,

(iii) $\gamma_n(ax) = a^n\gamma_n(x)$ for all $a \in A$, $x \in I$,

(iv) $\gamma_n(x)\gamma_m(y) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(xy)$ for all $x, y \in I$,

(v) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!m!^n}\gamma_{nm}(x)$ for all $x \in I$.

A **divided power ideal of** $\boldsymbol{A}$, or **PD-ideal**, is a tuple (I, γ) where $I \leq A$ is an ideal and and $\gamma = (\gamma_n)_n$ is a PD-structure on I. A **divided power algebra**, or **PD-algebra**, is a triple (A, I, γ) where A is a ring and (I, γ) is a PD-ideal of A.

A **morphism** of PD-algebras $(A, I, \gamma) \to (B, J, \delta)$ is a morphism of rings $f \colon A \to B$ such that $f(I) \subseteq J$ and such that $\delta_n f = f\gamma_n$ for all $n \geq 0$. ⌟

Definition 2.6. Let (A, I, γ) be a PD-algebra. Let B be an A-algebra and (J, δ) a PD-ideal in B. We say that γ and δ are **compatible**, if there is a PD-algebra structure $(B, IB, \overline{\gamma})$ together with a morphism of PD-algebras $(A, I, \gamma) \to (B, IB, \overline{\gamma})$ such that $\overline{\gamma}|_{IB\cap J} = \delta$. ⌟

We recall the two standard examples of PD-structures and PD-envelopes:

(i) Let A be a $\mathbb{Q}_p$-algebra and let $I \leq A$ be any ideal. The maps

$$\gamma_n \colon I \longrightarrow A, \qquad x \mapsto \frac{x^n}{n!}$$

define the unique PD-structure on I.

(ii) Let A be a $\mathbb{Z}_p$-algebra and let $I = (p)$. The maps

$$\gamma_n^{\mathrm{can}} \colon I \to A, \qquad pa \mapsto \frac{p^{n-1}}{n!}(pa^n)$$

define the **canonical PD-structure on (p)**.

(iii) Let (A, I, γ) be a PD-algebra and let B be an A-algebra with an ideal $J \leq B$. Then there exists a B-algebra $D_{B,\gamma}(J)$, called the **PD-envelope of (B, J) relative to (A, I, γ)**, with a PD-ideal $(\overline{J}, [-])$ such that $JD_{B,\gamma}(B) \subseteq \overline{J}$ and such that $[-]$ is compatible with γ. The PD-algebra $(D_{J,\gamma}(B), \overline{J}, [-])$ is universal among B-algebras C containing a PD-ideal (K, δ) such that $JC \subseteq K$ and such that δ is compatible with γ.

When working in characteristic p or mixed characteristic, instead of considering π-adic pro-infinitesimal thickening—as we did when defining $\mathbb{A}_{\mathrm{inf}}$—we now consider π-adic PD-thickenings for defining $\mathbb{A}_{\mathrm{cris}}$.

Definition 2.7. Let R be a p-complete $\mathbb{Z}_p$-algebra. A ***p*-adic PD-thickening of *R*** is a triple $(D, D \to R, (\gamma_n)_{n\geq 0})$ where D is p-complete, $D \to R$ is a surjection, and $(\gamma_n)_{n\geq 0}$ is a PD-structure on $J = \ker(D \to R)$ compatible with the canonical PD-structure on (p). ⌟

Definition 2.8. We define

$$\mathbb{A}_{\mathbf{cris}} \coloneqq D_{\mathbb{A}_{\mathrm{inf}},\gamma^{\mathrm{can}}}\big((\xi)\big)^{\wedge}_{p}$$

as the p-completion of the PD-envelope of $(\mathbb{A}_{\mathrm{inf}}, (\xi))$ relative to $(\mathbb{Z}_p, (p), \gamma^{\mathrm{can}})$. ⌟

Remark 2.9. One can show, that

$$\mathbb{A}_{\mathrm{cris}} \cong \mathrm{H}^0_{\mathrm{cris}}(\mathrm{Spec}(O_C)/\mathbb{Z}_p) \cong \mathrm{H}^0_{\mathrm{cris}}(\mathrm{Spec}(O_C/(p))/\mathbb{Z}_p)$$

where $\mathrm{H}^0_{\mathrm{cris}}(-/\mathbb{Z}_p)$ denotes crystalline cohomology. ⌟

Denote by $D_{\mathbb{Z}_p[x]}\big((x)\big)$ the PD-envelope of the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[x]$ and the ideal (x). The canonical morphism $\mathbb{Z}_p[x] \to \mathbb{A}_{\mathrm{inf}},\ x \mapsto \xi$ gives rise to an isomorphism

$$\mathbb{A}_{\mathrm{cris}} \cong \mathbb{A}_{\mathrm{inf}} \widehat{\otimes}_{\mathbb{Z}_p[x]} D_{\mathbb{Z}_p[x]}\big((x)\big)^{\wedge}_{p}.$$

Moreover,

$$D_{\mathbb{Z}_p[x]}\big((x)\big)^{\wedge}_{p} \cong \widehat{\bigoplus}_{n\geq 0} \mathbb{Z}_p \cdot \frac{x^n}{n!} \cong \big(\mathbb{Z}_p[y_0, y_1, y_2, \ldots]/(y_0 - x, y_1^p - py_0, y_2^p - py_1, \ldots)\big)^{\wedge}_{p}$$

is the free p-complete PD-algebra on one generator. In particular, we obtain

$$\mathbb{A}_{\mathrm{cris}}/(p) \cong O_C/(p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[y_1, y_2, \ldots]/(y_1^p, y_2^p, \ldots)$$

is a very non-noetherian and very non-perfect ring.

Denote by $\left[\frac{\xi^n}{n!}\right] \in \mathbb{A}_{\mathrm{cris}}$ the n-th divided power of ξ. Every element of $\mathbb{A}_{\mathrm{cris}}$ can be written as a power series

$$\sum_{n\geq 0} a_n \left[\frac{\xi^n}{n!}\right]$$

with $a_n \in \mathbb{A}_{\mathrm{inf}}$ converging to 0 in the p-adic topology. Note that this expression is not necessarily unique.

Definition 2.10. We define the **integral crystalline period ring** as

$$\mathbf{B}^{+}_{\mathbf{cris}} \coloneqq \mathbb{A}_{\mathrm{cris}}[p^{-1}]\,.$$

⌟

Remark 2.11. The ring $\mathrm{B}_{\mathrm{cris}}$, **Fontaine's ring of crystalline periods**, is constructed as a localisation of $\mathrm{B}_{\mathrm{cris}}$. ⌟

Lemma 2.12. *The natural morphism $\mathbb{A}_{\mathrm{inf}} \to \mathrm{B}^{+}_{\mathrm{dR}}$ extends to an injection*

$$\mathrm{B}^{+}_{\mathrm{cris}} \longrightarrow \mathrm{B}^{+}_{\mathrm{dR}}\,.$$

Proof. Note that $\mathrm{B}^{+}_{\mathrm{dR}}$ is a $\mathbb{Q}_p$-algebra by construction, hence the ideal (ξ) admits a unique PD-structure. Thus, the natural morphism $\mathbb{A}_{\mathrm{inf}} \to \mathrm{B}^{+}_{\mathrm{dR}}$ induces a morphism $D_{\mathbb{A}_{\mathrm{inf}},\gamma^{\mathrm{can}}}\big((\xi)\big) \to \mathrm{B}^{+}_{\mathrm{dR}}$ by definition. As $\mathrm{B}^{+}_{\mathrm{dR}}$ is complete with respect to the canonical topology and $\mathbb{A}_{\mathrm{cris}} = D_{\mathbb{A}_{\mathrm{inf}},\gamma^{\mathrm{can}}}\big((\xi)\big)^{\wedge}_{p}$, it suffices to show that the morphism

$$D_{\mathbb{A}_{\mathrm{inf}},\gamma^{\mathrm{can}}}\big((\xi)\big) \longrightarrow \mathrm{B}^{+}_{\mathrm{dR}}$$

is continuous with respect to the p-adic topology on the domain and the canonical topology on the codomain. Continuity can be checked on the finite level, so we consider the induced map

$$D_{\mathbb{A}_{\mathrm{inf}},\gamma^{\mathrm{can}}}\big((\xi)\big) \longrightarrow \mathbb{A}_{\mathrm{inf}}[p^{-1}]/(\xi^n)\,.$$

The image of this map is contained in $\frac{1}{(n-1)!}\mathbb{A}_{\mathrm{inf}}/(\xi^n)$ and since $\mathbb{A}_{\mathrm{inf}}/(\xi^n)$ is open by construction, this shows continuity. Thus, the natural morphism extends to a morphism

$$\mathbb{A}_{\mathrm{cris}} \longrightarrow \mathrm{B}^{+}_{\mathrm{dR}}$$

and as p is invertible in $\mathrm{B}^{+}_{\mathrm{dR}}$, we obtain a morphism

$$\mathrm{B}^{+}_{\mathrm{cris}} \longrightarrow \mathrm{B}^{+}_{\mathrm{dR}}\,.$$

It remains to check that this morphism is injective. Note that the axioms of the divided powers imply that

$$\xi^m\left[\frac{\xi^n}{n!}\right] = \frac{(n+m)!}{n!}\left[\frac{\xi^{n+m}}{(n+m)!}\right]\,.$$

In particular, if we can rewrite any finite sum

$$\sum_{n\geq 0}^{<\infty} a_n\left[\frac{\xi^n}{n!}\right] = \sum_{n\geq 0}^{<\infty} b_n\left[\frac{\xi^n}{n!}\right]$$

so that if $b_n \neq 0$, then $b_n \notin (\xi)$. By taking limits, we may assume that any element $x \in \mathbb{A}_{\mathrm{cris}}$ is of the form

$$x = \sum_{n\geq 0} a_n\left[\frac{\xi^n}{n!}\right]$$

so that if $a_n \neq 0$, then $a_n \notin (\xi)$. If $x \neq 0$, then there is a minimal n so that $a_n \neq 0$. In particular, there is a minimal n so that $a_n \notin (\xi)$. Thus, the ξ-adic valuation of x is finite and, hence, the image of x in $\mathrm{B}^{+}_{\mathrm{dR}}$ is non-zero. □

Appendix: The Rings of p-adic Hodge Theory

We briefly go over the rings appearing in the quote at the start.

Let us start with the *human rings*:

(i) $\mathbb{Q}_p$ is the field of p-adic numbers,

(ii) $\mathbb{Z}_p$ is the ring of p-adic integers, the ring of integers of $\mathbb{Q}_p$,

(iii) $\mathbb{F}_p$ is the finite field with p elements, the residue field of $\mathbb{Z}_p$,

(iv) $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$,

(v) $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$,

(vi) $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$,

(vii) $\mathcal{O}_{\mathbb{C}_p}$ is the ring of integers of $\mathbb{C}_p$,

(viii) $\mathbb{Q}_p^{\mathrm{unr}}$ is the maximal unramified subextension of $\overline{\mathbb{Q}}_p$, and

(ix) B_{HT} the *Hodge–Tate period ring*, is the ring of Laurent polynomials $\mathrm{B}_{\mathrm{HT}} = \mathbb{C}_p\left[t^{\pm}\right]$ over $\mathbb{C}_p$.

The *dwarvish rings* are the following:

(i) $\tilde{\mathrm{A}}$ is the ring of Witt vectors $\tilde{\mathrm{A}} = W(\mathbb{C}_p^\flat)$ of $\mathbb{C}_p^\flat$,

(ii) $\mathrm{E}_{\mathbb{Q}_p}$ is the subfield $\mathbb{F}_p(\!(\varepsilon - 1)\!)$ of $\mathbb{C}_p^\flat$ of norms of the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$,

(iii) $\mathrm{A}_{\mathbb{Q}_p}$ is the closure of $\mathbb{Z}_p\left[\pi^{\pm 1}\right]$, where $\pi = [\varepsilon] - 1$, in $\tilde{\mathrm{A}}$,

(iv) $\mathrm{B}_{\mathbb{Q}_p}$ is the ring $\mathrm{B}_{\mathbb{Q}_p} = \mathrm{A}_{\mathbb{Q}_p}[p^{-1}]$,

(v) E is a separable closure of E,

(vi) A is the unique p-saturated p-complete subring of $\tilde{\mathrm{A}}$ containing $\mathrm{A}_{\mathbb{Q}_p}$, and

(vii) B is the ring $\mathrm{B} = \mathrm{A}[p^{-1}]$.

Then there are the *elvish rings*:

(i) $\mathrm{B}_{\mathrm{cris}}$ is the *crystalline period ring*, as constructed in Section 2.2,

(ii) B_{st} is the *semistable period ring*, which as an abstract ring is $\mathrm{B}_{\mathrm{st}} = \mathrm{B}_{\mathrm{cris}}[u]$ for $u = \log\left(\frac{[p^\flat]}{p}\right)$, and

(iii) B_{dR} is the *de Rham period ring*, as constructed in Section 2.1.

A finally, of course, the *one ring to rule them all*:

(i) $\mathbb{A}_{\mathrm{inf}}$ is *Fontaine's first period ring*, as constructed in Section 1.

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