Locally Compact Abelian Groups

The main goal of this talk is to compute certain condensed Ext-groups of locally compact abelian groups in the derived category of condensed abelian groups and use these computations to compare Ext-groups of locally compact abelian groups to those of their condensed counterparts.

1 Introduction

A particular type of topological (abelian) groups we are interested in is the locally compact one. By this we mean nothing more than that the underlying topological is locally compact. A reminder:

Definition 1. A Hausdorff topological space X is ***locally compact*** if every point $x \in X$ has a compact neighbourhood (i.e. if there is some compact C and open U such that $x \in U \subseteq C$).

One interesting property of locally compact topological spaces is the behaviour of 'topological hom-sets'. The usual topology is the following.

Definition 2. Let X, Y be topological spaces. We call the topology generated by the subsets $V(C, U) = \{f \in \mathrm{Hom}(X, Y) \mid f(C) \subseteq U\}$ for all choices of compact $C \subseteq X$ and open $U \subseteq Y$ the ***compact-open topology*** on $\mathrm{Hom}(X, Y)$.

It is additionally guaranteed that the composition map $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is continuous if Y is locally compact.

Example 3. We provide some interesting (non)-examples of locally compact abelian groups:

(i) Discrete groups, e.g. $\mathbb{Z}$;

(ii) Local fields, e.g. $\mathbb{R}$ and $\mathbb{Q}_p$;

(iii) Compact groups, e.g $\mathbb{T} := \mathbb{R}/\mathbb{Z}$;

(iv) Two non-examples are $\mathbb{Q}$ and $\{(0,0)\} \cup \mathbb{R}_{>0} \times \mathbb{R}$.

We denote by **LCA** the category of locally compact abelian groups. We equip the continuous homomorphisms $\mathrm{Hom}_{\mathbf{LCA}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{Top}}(A, B)$ with the subspace topology.

Remark. The category **LCA** is not an abelian category, but it is a quasi-abelian. This means that it is additive, kernels and cokernels exists and these are stable under pushout and pullback respectively. However, if $f \colon A \to B$ is a morphism in **LCA**, we do not necessarily get an isomorphism $\mathrm{coker}(\ker f) \to \ker(\mathrm{coker}\, f)$. We call f *strict* if it does canonically induce such an isomorphism and we call an exact sequence *strict exact* if all of its morphisms are strict. The general idea is that many constructions in abelian categories are generalisable to quasi-abelian categories as long as we replace the notion of exactness by that of strict exactness.

It turns out that equipping a locally compact space with a commutative group structure is well-understood, as the following theorem shows.

Theorem 4. *Let A be a locally compact abelian group. We obtain an isomorphism $A \cong \mathbb{R}^n \times A'$, where A' is an extension of a discrete group by a compact one.*

Additionally, we have one more powerful tool at our disposal.

Definition 5. The ***Pontryagin dual*** of a locally compact abelian group A is $\mathbb{D}(A) = \mathrm{Hom}_{\mathbf{LCA}}(A, \mathbb{T})$.

Proposition 6. *Let A be a locally compact abelian group.*

(i) The topological abelian group $\mathbb{D}(A)$ is locally compact;

(ii) If A is discrete, then $\mathbb{D}(A)$ is compact and vice versa;

(iii) The natural map $A \to \mathbb{D}(\mathbb{D}(A))$ is an isomorphism.

Remark. Note the interesting examples $\mathbb{D}(\mathbb{R}) = \mathbb{R}$, $\mathbb{D}(\mathbb{Z}) = \mathbb{T}$ and $D(\mathbb{T}) = \mathbb{Z}$.

Corollary 7. *The category* **LCA** *is generated under filtered (co)limits, kernels, cokernels and extensions by $\mathbb{Z}$ and $\mathbb{T}$.*

Proof. Clearly $\mathbb{R}$ is an extension of $\mathbb{Z}$ and $\mathbb{T}$. Furthermore, any discrete group A admits a two-term free resolution $0 \to \bigoplus_I \mathbb{Z} \to \bigoplus_J \mathbb{Z} \to A \to 0$. Dually (by Proposition 6), any compact group A admits a two-term compact resolution $0 \to A \to \prod_I \mathbb{T} \to \prod_J \mathbb{T} \to 0$. Now Theorem 4 finishes the proof. ■

2 Passing to the condensed and derived setting

Our first observation when passing to the condensed setting is that the topological and condensed structures on hom-sets are often compatible in the following manner.

Proposition 8. *Let A, B be Hausdorff topological abelian groups, where A is additionally assumed to be compactly generated. Then*

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) = \underline{\mathrm{Hom}(A, B)}.$$

Proof. Note first that by $\mathrm{Hom}(A, B)$ we mean continuous *group homomorphisms*. We first construct natural maps $\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) \to \underline{\mathrm{Hom}_{\mathbf{Top}}(A, B)}(S)$ for all profinite sets S. We identify the left hand side with $\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) = \mathrm{Hom}_{\mathbf{Cond(Ab)}}(\mathbb{Z}[\underline{S}] \otimes \underline{A}, \underline{B})$ and the right hand side with

$$\begin{aligned}\underline{\mathrm{Hom}_{\mathbf{Top}}(A, B)}(S) &= \mathrm{Hom}_{\mathbf{Top}}(S, \mathrm{Hom}_{\mathbf{Top}}(A, B)) \\ &= \mathrm{Hom}_{\mathbf{Top}}(S \times A, B) \\ &= \mathrm{Hom}_{\mathbf{Cond(Set)}}(\underline{S \times A}, \underline{B}) \\ &= \mathrm{Hom}_{\mathbf{Cond(Ab)}}(\mathbb{Z}[\underline{S}] \otimes \mathbb{Z}[\underline{A}], \underline{B}).\end{aligned}$$

Precomposition with the (surjective) map $\mathbb{Z}[\underline{S}] \otimes \mathbb{Z}[\underline{A}] \to \mathbb{Z}[\underline{S}] \otimes \underline{A}$ induces an injection

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) \hookrightarrow \underline{\mathrm{Hom}_{\mathbf{Top}}(A, B)}(S).$$

To see what the image of this map is, we use the (partial) resolution

$$\mathbb{Z}[\underline{A} \times \underline{A}] \xrightarrow{\pi_1} \mathbb{Z}[\underline{A}] \xrightarrow{\pi_0} \underline{A} \to 0,$$

given on generators by maps $[(a_1, a_2)] \mapsto [a_1 + a_2] - [a_1] - [a_2]$ and $[a] \mapsto a$. By (left) exactness, a morphism $\mathbb{Z}[\underline{S}] \otimes \mathbb{Z}[\underline{A}] \to \underline{B}$ factors through $\mathbb{Z}[\underline{S}] \times \underline{A}$ if and only if it vanishes after precomposition with π_1. Identifying $\mathrm{Hom}_{\mathbf{Cond(Ab)}}(\mathbb{Z}[\underline{S}] \otimes \mathbb{Z}[\underline{A} \times \underline{A}], \underline{B}) = \mathrm{Hom}_{\mathbf{Top}}(S \times A \times A, B)$, one sees that precomposition with π_1 sends the map $f\colon S \times A \to B$ to the map $(s, a_1, a_2) \mapsto f(s, a_1 + a_2) - f(s, a_1) - f(s, a_2)$. Hence, the image of the map $\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) \hookrightarrow \underline{\mathrm{Hom}_{\mathbf{Top}}(A, B)}(S)$ is precisely $\underline{\mathrm{Hom}(A, B)}(S)$. ■

We can pass to the *derived* setting in two natural ways:

(1) We can stay in **LCA**. By replacing the notion of exactness by that of *strict* exactness one can still define a sensible derived category $D^\flat(\mathbf{LCA})$.

(2) We can pass from locally compact abelian groups to condensed groups via the functor $A \mapsto \underline{A}$, and then work in the category $D(\mathbf{Cond}(\mathbf{Ab}))$.

In fact, one can show that strict short exact complexes are mapped to exact complexes by said functor. In particular, we obtain a functor $D^\flat(\mathbf{LCA}) \to D(\mathbf{Cond}(\mathbf{Ab}))$. In the remainder of this talk we will see, by ways of computing Ext-groups and observing that all results agree, that this is really a fully faithful embedding.

3 Ext-groups

Let's first recall the construction of Ext-groups of objects X, Y in an abelian category $\mathcal{A}$ with enough projectives and injectives.

(1) Choose a projective resolution $P_\bullet \to X$. In the derived world, this amounts to choosing an isomorphic complex with projective entries.

(2) Apply $\mathrm{Hom}(-, Y)$ to obtain the complex $R\,\mathrm{Hom}(X, Y) = \mathrm{Hom}(P_\bullet, Y)$.

(3) Take (co)homology, i.e. $\mathrm{Ext}^i(X, Y) = H^i(\mathrm{Hom}(P_\bullet, Y))$.

Of course the same result can be achieved by replacing Y by an injective resolution.

This process yields in $\mathbf{Cond}(\mathbf{Ab})$ the objects $R\underline{\mathrm{Hom}}(X, Y) \in D(\mathbf{Cond}(\mathbf{Ab}))$ and $\underline{\mathrm{Ext}}^i(X, Y) \in \mathbf{Cond}(\mathbf{Ab})$. We specifically wish to compute the objects $R\underline{\mathrm{Hom}}(\underline{A}, \underline{B})$, where A, B are locally compact abelian groups.

To aid this computation, we introduce two computational tools. First, what we call the *Eilenberg-Maclane resolution*.

Theorem 9. *Let A be an abelian group. It has a functorial resolution*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A,$$

where $n_i, r_{i,j} \geq 0$ in **Ab**.

Corollary 10. *Let $\mathcal{A}$ be a condensed abelian group. It has a resolution*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathcal{A}^{r_{i,j}}] \to \cdots \to \mathbb{Z}[\mathcal{A}^3] \oplus \mathbb{Z}[\mathcal{A}^2] \to \mathbb{Z}[\mathcal{A}^2] \to \mathbb{Z}[\mathcal{A}] \to \mathcal{A}$$

in $\mathbf{Cond}(\mathbf{Ab})$*, where $n_i, r_{i,j} \geq 0$.*

Proof. This follows from functoriality and the exactness of sheafification. ∎

Corollary 11. *Let $\mathcal{A}, \mathcal{M}$ be condensed abelian groups and let S be an extremally disconnected set. There is a spectral sequence*

$$E_1^{i_1 i_2} = \prod_{j=1}^{n_{i_1}} H^{i_2}(\mathcal{A}^{r_{i_1,j}} \times \underline{S}, \mathcal{M}) \implies \underline{\operatorname{Ext}}^{i_1+i_2}(\mathcal{A}, \mathcal{M})(S)$$

that is functorial in $\mathcal{A}, \mathcal{M}$ and S.

Proof. Resolve $\mathcal{A}$ as in Corollary 10 and tensor with $\mathbb{Z}[\underline{S}]$, which is exact. Now apply $R\operatorname{Hom}(-, \mathcal{M})$ and note that $H^i(R\operatorname{Hom}(\mathbb{Z}[\mathcal{A}^r \times \underline{S}], \mathcal{M})) = H^i(\mathcal{A}^r \times \underline{S}, \mathcal{M})$. The proof now follows from the fact that

$$\begin{aligned} \underline{\operatorname{Ext}}^i(\mathcal{A}, \mathcal{M})(S) &= H^i(R\underline{\operatorname{Hom}}(\mathcal{A}, \mathcal{M}))(S) \\ &= H^i(R\underline{\operatorname{Hom}}(\mathcal{A}, \mathcal{M})(S)) \\ &= H^i(R\operatorname{Hom}_{\mathbf{Cond(Ab)}}(\mathbb{Z}[\underline{S}], R\underline{\operatorname{Hom}}(\mathcal{A}, \mathcal{M}))) \\ &= H^i(R\operatorname{Hom}_{\mathbf{Cond(Ab)}}(\mathbb{Z}[\underline{S}] \otimes \mathcal{A}, \mathcal{M})). \end{aligned}$$ ∎

To compare *all* Ext-groups of locally compact abelian groups with those of their corresponding condensed abelian groups, it suffices to do so for $R\underline{\operatorname{Hom}}(\underline{A}, \underline{B})$ with $A, B \in \{\bigoplus \mathbb{Z}, \prod \mathbb{T}, \mathbb{R}\}$ by Corollary 7. In fact, since $\mathbb{Z} = \mathbb{Z}[\underline{*}]$ is projective and the case $A = \mathbb{R}$ follows from $A = \mathbb{Z}$ and $A = \mathbb{T}$, it suffices to consider $A = \prod \mathbb{T}$. Similarly in the second argument, the case $B = \mathbb{T}$, therefore also $B = \prod \mathbb{T}$ follows from $B = \mathbb{R}$ and $B = \mathbb{Z}$. Hence, we only need to explicitly compute the cases where $A = \prod \mathbb{T}$ and B is either discrete or $\mathbb{R}$. We obtain the following results:

Theorem 12. *Let I be a (possibly infinite) set. Write $A = \prod_I \mathbb{T}$.*

(i) If B is discrete, then

$$R\underline{\operatorname{Hom}}(\underline{A}, \underline{B}) = \bigoplus_I \underline{B}[-1],$$

where the map $\bigoplus_I \underline{B}[-1] \to R\underline{\operatorname{Hom}}(\underline{A}, \underline{B})$ is induced by the maps

$$\underline{B}[-1] = R\underline{\operatorname{Hom}}(\underline{\mathbb{Z}}[1], \underline{B}) \to R\underline{\operatorname{Hom}}(\underline{\mathbb{T}}, \underline{B}) \xrightarrow{p_i^*} R\underline{\operatorname{Hom}}(\underline{A}, \underline{B}),$$

where $p_i \colon A \to \mathbb{T}$ is the i-th projection.

(ii) If $B = \mathbb{R}$, then

$$R\underline{\operatorname{Hom}}(\underline{A}, \underline{B}) = 0.$$

Proof. (i) We deal with the case where I is finite first. Since $\prod_I \mathbb{T} = \bigoplus_I \mathbb{T}$ now, it suffices to show that $R\underline{\operatorname{Hom}}(\underline{\mathbb{Z}}[1], \underline{B}) \to R\underline{\operatorname{Hom}}(\underline{\mathbb{T}}, \underline{B})$ is an isomorphism i.e. that $R\underline{\operatorname{Hom}}(\underline{\mathbb{R}}, \underline{B}) = 0$. To achieve this, we show that $0 \to \mathbb{R}$ induces an isomorphism on $\underline{\operatorname{Ext}}$-groups. By Corollary 11, it suffices to prove that the map (induced by $0 \to \mathbb{R}$) $H^i(\mathbb{R}^r \times S, B) \to H^i(S, B)$ is an isomorphism for any profinite S and $r \geq 0$. This follows from the fact that

$H^i([-n,n]^r \times S, B) \to H^i(S,B)$ is an isomorphism (using the fact that sheaf cohomology is homotopy invariant) by taking the derived limit.

Let I now be infinite. We just need to show that the colimit over all finite subsets of I induces an isomorphism

$$\mathrm{colim}_{J\subseteq I}\, R\underline{\mathrm{Hom}}(\underline{\prod_J \mathbb{T}}, \underline{B}) = R\underline{\mathrm{Hom}}(\underline{A}, \underline{B}).$$

By Corollary 11, it suffices to check this fact on (Čech) cohomology groups, where this is a known fact (for example by a result in the previous talk).

(ii) (*Sketch*) Write $F(A)_\bullet$ for the Eilenberg-Maclane resolution of $\underline{A}$ from Corollary 10. Now $R\underline{\mathrm{Hom}}(A,\mathbb{R})(S)$ is computed by the complex

$$0 \to \bigoplus_{j=1}^{n_0} C(A^{r_{0,j}} \times S, \mathbb{R}) \to \bigoplus_{j=1}^{n_1} C(A^{r_{1,j}} \times S, \mathbb{R}) \to \dots,$$

where now $C(X,Y) = \mathrm{Hom}_{\mathbf{Top}}(X,Y)$. The crucial idea in this proof is now that multiplication by 2 is now bounded on $A = \prod_I \mathbb{T}$, but not on $\mathbb{R}$. We use the fact that the maps $\cdot 2$ and $[2]$ (induced by multiplication by 2 on A) are in fact homotopic via a homotopy $h_\bullet \colon F(A)_\bullet \to F(A)_{\bullet+1}$. We show now that the above complex is exact, hence take some $f \in \bigoplus_j^{n_i} C(A^{r_{i,j}} \times S, \mathbb{R})$ such that $df = 0$. Then $2f - [2]^*f = d(h_{i-1}^*(f))$, and similarly for all $n \geq 0$ we find

$$f = \frac{1}{2^n}[2^n]^*(f) + d(\frac{1}{2}h_{i-1}^*(f) + \frac{1}{4}h_{i-1}^*([2]^*(f)) + \dots + \frac{1}{2^n}h_{i-1}^*([2^{n-1}]^*(f))).$$

Now, since the image of $[2^n]^*(f)$ stays bounded and h_{i-1}^* has bounded norm, we get after taking the limit $n \to \infty$ that $f \in \mathrm{Im}\, d$. ■

We conclude by showing that the passage from locally compact groups to condensed abelian groups is fully faithful.

Corollary 13. *The functor $D^\flat(\mathbf{LCA}) \to D(\mathbf{Cond}(\mathbf{Ab}))$ from Section 2 is fully faithful.*

Proof. It suffices to show that for any two locally compact abelian groups A, B, $R\,\mathrm{Hom}_{\mathbf{LCA}}(A,B) \to R\,\mathrm{Hom}(\underline{A}, \underline{B})$ is an isomorphism. This follows from Theorem 12, its preceding discussion and the corresponding calculations on $D^\flat(\mathbf{LCA})$ by Hoffmann and Spitzweck. ■

Remark. Note in particular that all $\underline{\mathrm{Ext}}^i(\underline{A}, \underline{B}) = 0$ for $A, B \in \mathbf{LCA}$ and $i \geq 2$.