Algebra seminar
2 Jul 25

Families of elliptic curves

non-algebraicity of solutions of diff. eq. associated to families of elliptic curves

Lam, Litt: arXiv 2501.13175, sec. 4

goal: show that conjecture of Lam-Litt holds for Gauß-Manin connection / Picard-Fuchs equations of

$\boxed{\text{non-isomonodromic families of elliptic curves}}$

Conjecture (Lam-Litt, conj. 2.3.1. in paper)

$R \subseteq \mathbb{C}$ integral domain, finitely gen^d $/\mathbb{Z}$

$(X, \mathcal{F})$ sm. scheme w/ foliation $/R$. $x \in X(R)$

TFAE

1. the analytic leaf of $\mathcal{F}$ through $x_{\mathbb{C}}$ is algebraic
2. the formal leaf through x is integral

[3. the leaf through x is $w(p)$-integral]

omitted in seminar, weakening of integrality, control denominators

Remark: for vb w/ connection $(\mathcal{E}, \nabla)$

get foliation on $\mathbb{V}(\mathcal{E})$, leaves are graphs of flat sections / ODE solutions

$z \longmapsto (z, f(z), f'(z), \dots, f^{(d-1)}(z))$

this is the situation we consider (as in Erik's talk)

§ 1 setup & statement of theorem

arithmetic
base scheme: $R \subseteq \mathbb{C}$ f.g. $\mathbb{Z}$-alg, $k = \mathrm{Frac}(R)$

$S / \operatorname{Spec} R$ sm, connected,

e.g.: $R = \mathbb{Z}[\frac{1}{N}]$ $S = \mathbb{P}^1_R \setminus \{0, 1, \infty\}$

note: only finitely many pts Siegel thm unit equation.

also, no ell. curves over $\mathbb{Z}$:

families of elliptic curves: $f : \mathcal{X} \to S$

sm, proj, relative dim 1, geom. fibers genus 1 curves,

non-isomonodromic: $\operatorname{Jac}(\mathcal{X}/S)$ has nonconst j-invariant

we'll look at examples in §2, j-inv vs. monodromy in §4

Gauß-Manin connection: $(\mathcal{E}, \nabla) = (R^1 f_* \Omega^*_{dR, \mathcal{X}/S}, \nabla_{GM})$

part of cohomology of $\mathrm{Sym}^n(\mathcal{X}/S)$

Theorem (Lam-Litt, thm 4.1.1 in paper)

in this setting, consider vb w/ connection $\mathrm{Sym}^n_{n>0}(\mathcal{E}, \nabla)$

initial conditions:

$$s \in S(R), \quad 0 \neq v \in \mathrm{Sym}^n H^1_{dR}(\mathcal{X}_s)$$

$$\boxed{\text{Then the formal flat section of } \mathrm{Sym}^n(\mathcal{E}, \nabla)_k \text{ through } v_k \text{ is not integral (resp. not } w(p)\text{-integral)}}$$

Remark: plan: discuss background material & example in §2, §3

show nonalgebraicity of sections in §4
then §5 pf of 4.1.1.
} shows conjecture 2.3.1 is true in this case.

§2 background elliptic curves & examples

elliptic curve: smooth proj. genus 1 curve $E/_K$
w/ choice of K-rational pt $0 \in E(K)$.

e.g. $y^2 = f(x)$, $\deg f = 3$, no multiple zeros, then homogenize for projective.

over $\mathbb{C}$: $E \simeq \mathbb{C}/_{\Lambda}$ for lattice $\underset{\underset{\mathbb{Z}^2}{\wr\!\wr}}{\Lambda} \leq \mathbb{C}$

$$\dim H^i_{dR}(E/_{\mathbb{C}}) = \begin{cases} 1 & i = 0,2 \\ 2 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\leftarrow$ uninteresting
$\leftarrow$ very interesting (why we restrict to $R^1 f_*$ in Thm 4.1.1.)

Hodge theory:

$$H^1_{dR}(E/\mathbb{C}) \simeq H^{0,1} \oplus H^{1,0}$$

$$F^1_{Hodge} H^1_{dR}(E) = H^0(E, \Omega^1_{E/K}) \leq H^1_{dR}(E/_K)$$

$\llcorner$ global diff forms

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now consider families $f: \mathcal{E} \to S$ as in Thm 4.1.1:

Example: Legendre family

ell. curves $y^2 = x(x-1)(x-\lambda)$ over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ (parameter λ)

invariant differential

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

$S = \mathbb{P}^1_R \setminus \{0,1,\infty\}$
$R = \mathbb{Z}[\frac{1}{N}]$

generates $H^0(E, \Omega^1)$

fix pt on E, e.g. 0

then $p \longmapsto \int_\gamma \omega$ defines $E \longrightarrow \mathbb{C}/_{\Lambda}$

γ path $0 \leadsto p$

depends on path
$\sim$ monodromy!

Λ period lattice $\left\{ \int_\gamma \omega \;\middle|\; \underset{\text{closed paths}}{\gamma \in \pi_1(E,0)} \right\}$

$\simeq H_1(E, \mathbb{Z})$

Historical note: $\int \frac{(1-cx)\,dx}{\sqrt{x(1-x)(1-cx)}}$ compute arc length of ellipse
$\rightsquigarrow$ elliptic integral $\rightsquigarrow$ elliptic curve
$\rightsquigarrow$ abelian integrals $\rightsquigarrow$ abelian varieties

Similar integral appears in computing period of oscillation of mechanical systems (planets, pendulum, ...) $\rightsquigarrow$ periods

further examples:

Rmk: for Legendre family
$$S = \operatorname{Spec} \mathbb{Z}\left[\lambda, \frac{1}{\lambda(\lambda-1)}\right] = Y(2)$$
coarse moduli space for ell. curves w/ level-2 structure (suitable choice of 2-torsion pts)

- other example:
$$Y(3) = \operatorname{Spec} \mathbb{Z}\left[\zeta_3, \tfrac{1}{3}, t, \frac{1}{t^3-1}\right]$$
$\sim$ general example $Y(N)$

family $x^3+y^3+z^3 = 3t\,xyz$ (Hesse cubic)

- example in lit.: Stienstra–Beukers, cf also Zagier ell. modular forms
$$x(y-z)(z-x) - \tau(x-y)yz = 0$$
related to Apéry's recursion in irrationality pf's

Picard–Fuchs eq.
$$\tau(\tau^2-11\tau-1)f'' + (3\tau^2-22\tau-1)f' + (\tau-3)f = 0$$

- $f(z)$ modular form of wt k, $t(z)$ modular function, both for gp Γ,

locally $f(z) = \Phi(t(z))$ then $\Phi(t)$ satisfies linear ODE of order $k+1$ w/ alg coeffs.

$\rightsquigarrow$ many more examples also involving $\mathrm{Sym}^n(\mathcal{E},\nabla)$

e.g. Legendre family
$$f = \theta_3(z)^2, \quad \text{wt } 1 \text{ modular form}$$
$$t = \lambda(z) \quad \text{Legendre modular function}$$

or expansion of Eisenstein series $f = E_4(z)$ over $t(z) = \frac{1728}{j(z)}$

§3 Picard-Fuchs equation & solutions

Picard-Fuchs equation

(description of ∇_{GM} on $R^1 f_* \Omega^1_{\mathcal{E}/S}$)

recall $H_1(E,\mathbb{Z})$ has rk 2, $H^1_{dR}(E)$ has rk 2

sections of $R^1 f_* \Omega^1_{\mathcal{E}/S}$: ω, $\omega' := \nabla_{GM}(\omega) = \dfrac{dx}{2\sqrt{x(x-1)(x-\lambda)^3}}$

$\rightsquigarrow$ express $\omega'' := \nabla_{GM}(\omega') = \dfrac{3\,dx}{4\sqrt{x(x-1)(x-\lambda)^5}}$ derivatives of ω wrt λ

as linear comb (over $\mathcal{O}(\mathbb{P}^1 \setminus \{0,1,\infty\}) = \mathbb{Z}[\lambda, \frac{1}{\lambda(\lambda-1)}]$) of ω, ω'

$$\boxed{\lambda(\lambda-1)\,\omega'' + (2\lambda-1)\,\omega' + \frac{1}{4}\omega = 0}$$

Picard-Fuchs equation for Legendre family

2nd order linear ODE

$\omega^{(n)}$ appear linearly

Rem: historical origin: the periods $\int_\gamma \omega$, $\gamma \in H_1(E,\mathbb{Z})$ also satisfy this equation

Rem:

rewrite $$\omega'' + \underbrace{\frac{(2\lambda-1)}{\lambda(\lambda-1)}}_{a(\lambda)}\,\omega' + \underbrace{\frac{1}{4\lambda(\lambda-1)}}_{b(\lambda)}\,\omega = 0$$

then limits $\lim_{\lambda \to \text{sing pt}} \lambda a(\lambda)$, $\lim_{\lambda\to\text{sing pt}} \lambda^2 b(\lambda)$ exist, e.g.

$$\lim_{\lambda\to 0} \lambda a(\lambda) = 1 \qquad \lim_{\lambda\to 0} \lambda^2 b(\lambda) = 0$$

$\Longrightarrow$ regular singularities @ $0, 1, \infty$:

Solutions: Taylor expansion & recursions

Taylor expansion around $\lambda = a$:

power series Ansatz: $\omega(\lambda) = \sum_{n\geq 0} c_n(\lambda-a)^n$, plug into Picard-Fuchs eq.

$$((\lambda-a)+a)((\lambda-a)+a-1)\sum_{n\geq 2} n(n-1)c_n(\lambda-a)^{n-2}$$

$$(2(\lambda-a)+2a-1)\sum_{n\geq 1} nc_n(\lambda-a)^{n-1} + \frac{1}{4}\sum_{n\geq 0} c_n(\lambda-a)^n =$$

$$= \sum_{n\geq 2} n(n-1)c_n(\lambda-a)^n + (2a-1)\sum_{n\geq 2} n(n-1)c_n(\lambda-a)^{n-1}$$

$$+ a(a-1)\sum_{n\geq 2} n(n-1)c_n(\lambda-a)^{n-2}$$

$$+ 2\sum_{n\geq 2} nc_n(\lambda-a)^n + (2a-1)\sum_{n\geq 1} nc_n(\lambda-a)^{n-1}$$

$$+ \frac{1}{4}\sum_{n\geq 0} c_n(\lambda-a)^n = 0$$

$\rightsquigarrow$ recursion for coefficients c_n initial condition c_0, c_1 fixed.

$$\boxed{\begin{aligned} &a(a-1)(n+1)(n+2)\,c_{n+2} \\ &+ (2a-1)(n+1)^2 c_{n+1} + \left(n+\tfrac{1}{2}\right)^2 c_n = 0 \end{aligned}}$$

Thm 4.1.1. implies that for any $a \neq 0, 1$ and any initial conditions c_0, c_1 (not both 0!)

- (c_n) near have so many primes in denominators. /
- (c_n) near do not lie in f.g. $\mathbb{Z}$-alg.

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expl: $a = \frac{1}{2}$ is simpler:

$$c_{n+2} = -\frac{(n+\frac{1}{2})^2 c_n}{4(n+1)(n+2)} = -\frac{(2n+1)^2 c_n}{16(n+1)(n+2)}$$

for even c_{2n}, primes $p \equiv 3 \bmod 4$ appear in denominator of c_{p+1},

$\rightsquigarrow$ not integral, not even ω-p-integral

Taylor expansion around $\lambda = 0$: (regular singular pt)

Frobenius method: indicial equation $r(r-1) + a_0 r + b_0 = 0$

for $a_0 = \lim_{\lambda \to 0} \lambda a(\lambda)$, $b_0 = \lim_{\lambda \to 0} \lambda^2 b(\lambda)$

in our case $r^2 = 0$

factor $(\lambda)^r = 1$!

$\rightsquigarrow$ solutions of the form

$$w_1(\lambda) = \sum_{n \geq 0} c_n \lambda^n \qquad c_0 = 1$$

$$w_2(\lambda) = \sum_{n \geq 0} d_n \lambda^n + w_1(\lambda) \log(\lambda) \qquad d_0 = 1.$$

as above, plug into eq

$\rightsquigarrow$ recursion $c_1 = \frac{1}{4} c_0$, $4c_2 = \frac{9}{64} c_1$, $c_{n+1} = \frac{(n+\frac{1}{2})^2}{(n+1)^2} c_n$

$$\boxed{w(\lambda) = \sum_{n \geq 0} \frac{1}{16^n} \binom{2n}{n}^2 \lambda^n}$$

for $\lambda = 1$ same expansion w/ (-1) factor.

$= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; z\right)$

$=$ expansion of $\theta_3(z)^2$ in terms of $\lambda(z)$ Legendre modular function,

for $\theta_3 = \sum q^{\frac{n^2}{2}}$ Jacobi θ function

also connected to point counts on fibers, Dwork's work on zeta functions.

NB: this solution has only powers of 2 in denominator ↯

but Thm. 4.1.1. / Conj. 2.3.1. doesn't apply

expansion around regular singular pt $\rightsquigarrow$ initial conditions don't make sense

Rmks: what has Taylor expansion around $\lambda = 0$

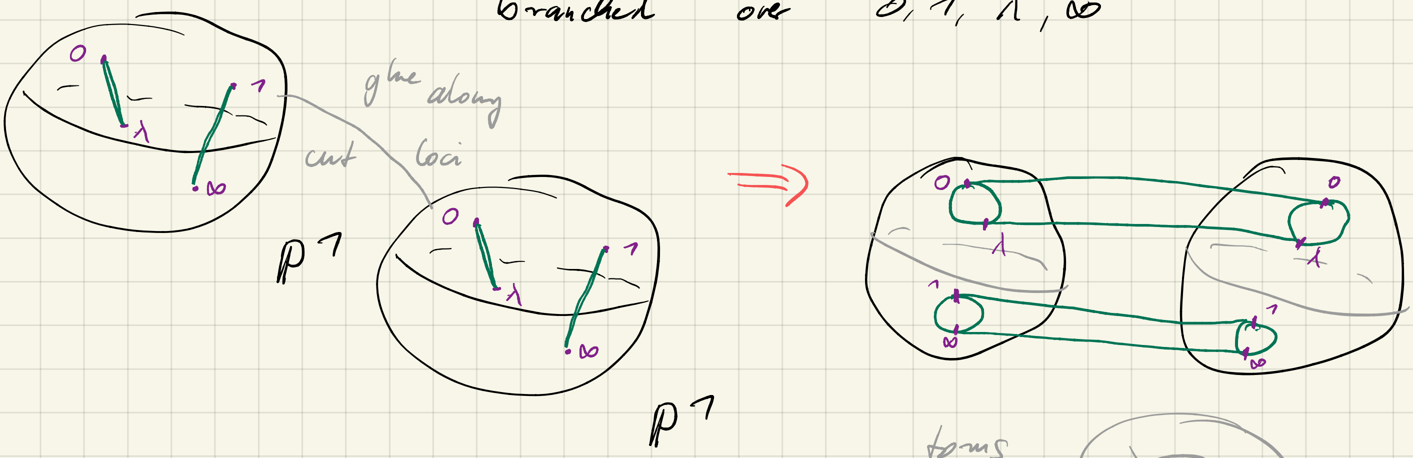
have to do w/ vanishing / nearby cycles?

are these concepts relevant for the conjecture?

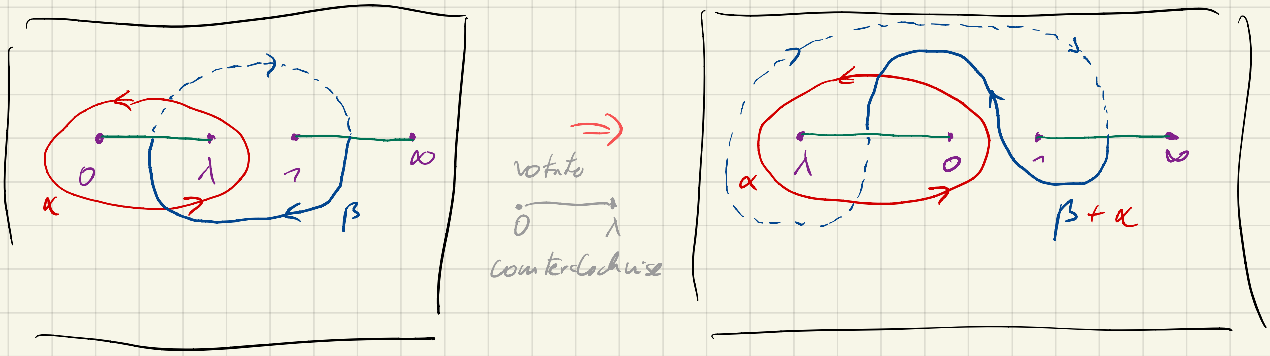
§ 4 monodromy & non-algebraicity of solutions

Monodromy of Legendre family:

fiber at λ: elliptic curve as deg 2 ramified covering branched over $0, 1, \lambda, \infty$



generators of homology $H_1(E;\mathbb{Z})$:



computing monodromy:

full loop of λ around 0: monodromy $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$

analogous monodromy around 1: $\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$

$\rightsquigarrow$ monodromy gp $\Gamma(2) = \ker\left(SL_2\mathbb{Z} \xrightarrow[\text{mod } 2]{} SL_2\mathbb{F}_2\right)$

index 6 subgp. in $SL_2\mathbb{Z}$.

Zariski closure in $\mathrm{Aut}(H_1(E,\mathbb{C})) = GL_2(\mathbb{C})$

$\rightsquigarrow$ algebraic monodromy SL_2.

analogous for universal families over $Y(N) = \mathbb{H}/\Gamma(N)$ monodromy $\Gamma(N)$

non-constant j-function implies monodromy SL_2:

(Deligne: equidistribution result in Weil II, 3.5.5.)

j-function: modular function, provides identification $\mathbb{H}/_{SL_2\mathbb{Z}} \xrightarrow{\simeq} \mathbb{C}$ — isomorphism classification of ell. curves $/\mathbb{C}$.

Prop: family of ell. curves $f: \mathfrak{X} \to S$ $/\mathbb{C}$

$j: S \to \mathbb{C}$ nonconst $\Rightarrow$ f has monodromy SL_2

pf sketch: pass to finite étale cover $\tilde{S}/S$ to trivialise N-torsion

$\rightsquigarrow$ morphism $\varphi: S \to Y(N)$ classifying $f: \mathfrak{X} \to S$

j-function nonconst $\Rightarrow$ φ dominant

monodromy claim reduces to $Y(N)$ case, see above. ∎

Rem: for Lam-Litt paper monodromy SL_2 is the relevant thing, nonconst j-function one way to get that (& easier to compute j-fn. than computing monodromy)

$\Rightarrow$ in the situation of Thm 4.1.1.,

monodromy rep $SL_2 \curvearrowright H^1_{dR}(\mathfrak{X}_s)$ is irreducible

obvious matrix rep.

from rep. theory of reductive gps:

$SL_2 \curvearrowright V$ defining 2-dim repr.

then irreducible representations of SL_2 given by $\mathrm{Sym}^n(V)$.

nonalgebraicity of solutions:

for Legendre family: can use Eisenstein thm

solution $\varphi \in \mathbb{Q}[[z]]$ alg & then apply Thm 4.1.1.

$\Rightarrow \quad \varphi \in \mathbb{Z}[\frac{1}{N}][[z]]$ almost integral.

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in general: situation of Thm 4.1.1. implies

monodromy rep on $R^1 f_* \Omega^q_{X/S}$ is irreducible

Prop: irreducible monodromy + algebraic solution

$\Rightarrow$ monodromy finite

Pf sketch: solution φ algebraic $\Leftrightarrow$ orbit under monodromy is finite

$\Rightarrow$ translates of φ under monodromy are algebraic

monodromy irreducible $\Rightarrow$ full set of alg. solutions $\Rightarrow$ monodromy finite.

Cor: In the situation of Thm 4.1.1, no nonzero solution is algebraic.

Cor: In the situation of Thm 4.1.1, Thm 4.1.1 implies Conjecture 2.3.1.

Rem: – argument for nonalgebraicity works in many other settings: Hilbert modular forms or families of abelian varieties (characterization of irred. monodromy more complicated than in elliptic curve case, exclude CM)

– connection to transcendence of numbers?

§5 pf of Theorem 4.1.1

pf steps: $\tilde{v}_K$ formal flat section of $\mathrm{Sym}^n(\mathcal{E},\nabla)_K$ through v_K

assume $\tilde{v}_K$ is integral,
i.e. can reduce mod p for almost all primes

focus on special case $n=1$, i.e. loc. sys of rk 2,
& 1-dim base S over $R = \mathbb{Z}[\frac{1}{N}]$

(1) p-curvature vanishes along formal flat section $\tilde{v}_K$ mod p.

cf §6 p-curvature recap

(2) find ∞ many primes p (Zariski dense set of closed pts of R)

s.th. v mod p not contained in F^1_{conj}

$\rightsquigarrow$ p-curvature vanishes on $gr^0_{conj}(\mathcal{E},\nabla)$

cf §7 conjugate filtration recap

(3) by theorem of Katz (next page)

∇_{GM} vanishes on $gr^1_{Hodge}(\mathcal{E},\nabla)$ at s

(mod p for ∞ many primes & therefore in char 0 also!)

$\rightsquigarrow$ ∇_{GM} preserves $F^1_{Hodge}(\mathcal{E},\nabla)$

$\rightsquigarrow$ monodromy repr. is reducible,
i.e. has 1-dim subrepresentation

contradicts assumption of Thm 4.1.1

Rem: changes for Sym^n, $n>1$,
above argument works for some step of Hodge filtration

Rem.: changes for $w(p)$ - integrability

can only reduce $w(p)$ - jets, still implies

vanishing of p-curvature to order $w(p)$, then ∇_{GM} vanishes

to order $w(p)/p$. $\lim_{p\to\infty} \frac{w(p)}{p} = \infty$ implies vanishing of ∇_{GM}

in char 0.

recall from Doosung's talk:

Theorem (Katz '72 Thm 3.2, Thm 4.2.1 in lecture Litt)

$f: X \to S$ sm. proj. / field k of char $p > 0$.

assume $R^a f_* \Omega^b_{X/S}$ loc. free, compatible w/ base change

& Hodge-to-de Rham spectral seq. deg. at E_1

then conj. spectral seq. deg at E_2 & we have comm. diag.

$$\begin{tikzcd}
F_{abs}^* \operatorname{gr}^b_{Hodge} R^{a+b} f_* \Omega^\bullet_{X/S} \arrow[r, "F_{abs}^* \operatorname{gr}_{Hodge} \nabla_{GM}"] \arrow[d, "\simeq"', "\text{deg, base change \& inverse Cartier op. } C^{-1}"] & F_{abs}^* \operatorname{gr}^{b-1}_{Hodge} R^{a+b} f_* \Omega^\bullet_{X/S} \otimes F_{abs}^* \Omega^1_{S/k} \arrow[d, "\simeq"] \\
\operatorname{gr}^a_{conj} R^{a+b} f_* \Omega^\bullet_{X/S} \arrow[r, "-\operatorname{gr}_{conj} \psi_p"'] & \operatorname{gr}^{a+1}_{conj} R^{a+b} f_* \Omega^\bullet_{X/S} \otimes F_{abs}^* \Omega^1_{S/k}
\end{tikzcd}$$

Rem.:

$\operatorname{gr}_{Hodge} \nabla_{GM}$: Hodge-graded of Gauß-Manin conn.

degree -1 by Griffiths transversality

$\operatorname{gr}_{conj} \psi_p$: induced by p-curvature

deg $+1$ because p-curvature vanishes

on assoc. graded of conj. filtration by results of Katz for step ③ above, apply this for mod p fibs of $f: \mathcal{X} \to S$ $/_R$ w/ $b = 1$, $a = 0$

Rem:

① conditions of thm are satisfied because our situation lifts to char. 0.

② $F_{abs}: S \to S$ faithfully flat, (smoothness of S + theorem of Kunz)

$\rightsquigarrow$ vanishing of $F_{abs}^* \, gr_{Hodge} \nabla_{GM}$ implies vanishing of $gr_{Hodge} \nabla_{GM}$

③ view $gr_{Hodge} \nabla_{GM}$ as morphism of coherent sheaves — vanishing mod p for Zariski dense set of p implies vanishing

[note we are not using $\mathcal{O}_S$-linearity which we don't have, we only use R-linearity !]

Rem: is there a more direct way to connect p-curvature w/ same monodromy in char p?

maybe holonomy is the better way to think about obstruction to p-curvature vanishing?

is this related to Katz' conjecture relating monodromy & differential Galois gp?

§6 p-curvature map

recall definition of p-curvature (from Vivian's talk)

S smooth / field k of char $p>0$, $(\mathcal{E},\nabla)$ vb w/ flat / integrable connection

view connection as

$$\nabla : \underset{\substack{\rotatebox{90}{$\cong$} \\ T_{S/k}}}{\mathrm{Der}(S/k)} \longrightarrow \mathrm{End}_k(\mathcal{E}) : \quad D \in \mathrm{Hom}(\Omega^1, \mathcal{O}) \longmapsto \left(\nabla_D : \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{S/k} \xrightarrow{1 \otimes D} \mathcal{E}\right)$$

Def: p-curvature of connection ∇

$$\boxed{\psi_p : \mathrm{Der}(S/k) \longrightarrow \mathrm{End}_S(\mathcal{E}) : \quad D \longmapsto (\nabla_D)^p - \nabla_{(D^p)}}$$

improved linearity from p-th powers in char p.

p-curvature for flat connections vs foliations (from Erik's talk)

$(\mathcal{E},\nabla)$ vb w/ flat connection $/S$,

$\mathbb{V}(\mathcal{E}) = \mathrm{Spec}_S(\mathrm{Sym}^\bullet \mathcal{E}^\vee)$ total space of $\mathcal{E}$ as S-scheme, $\pi : \mathbb{V}(\mathcal{E}) \to S$

$\rightsquigarrow$ exact seq. of tangent bdls

$$0 \longrightarrow \underset{\substack{\rotatebox{90}{$\cong$} \\ \pi^*\mathcal{E}}}{T_{\mathbb{V}(\mathcal{E})/S}} \longrightarrow T_{\mathbb{V}(\mathcal{E})/k} \longrightarrow \pi^* T_{S/k} \longrightarrow 0$$

vertical tangent bdl

section σ from ∇

defines foliation $\mathcal{F} = \mathrm{Im}\, \sigma$

(because ∇ flat)

$$\boxed{\text{formal leaves for } \mathcal{F} = \text{formal flat sections of } (\mathcal{E},\nabla)}$$

p-curvature for $(\mathcal{E},\nabla)$: $\psi_p : T_{S/k} \longrightarrow \mathrm{End}_S(\mathcal{E})$

p-curvature for $\mathcal{F}$: $\psi_p : \underset{\substack{\rotatebox{90}{$\cong$} \\ F_{abs}^* \pi^* T_{S/k}}}{F_{abs}^* \mathcal{F}} \longrightarrow T_{\mathbb{V}(\mathcal{E})/k} \longrightarrow \underset{\substack{\rotatebox{90}{$\cong$} \\ \pi^*\mathcal{E}}}{T_{\mathbb{V}(\mathcal{E})/k} / \mathcal{F}}$

according to Lam-Litt Def 2.4.2 (following Bost)

Rem: under identifications, p-curv. for foliation should be pullback

of p-curv for $(\mathcal{E},\nabla)$ along $\pi: V(\mathcal{E}) \to S$,

F^*_{abs} deals w/ p-power in def of p-curv.

I'm unsure about $\mathcal{O}_X$-linearity claim in def of Car. Lift.

$\hat{=}$ $\mathrm{End}_S \mathcal{E}$ vs $\pi^*\mathcal{E}$ as target - problem

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for the rest, we work w/ p-curvature for $(\mathcal{E},\nabla)$

Cartier's thm as Frobenius thm in char $p>0$

char 0: distribution — k-plane subbdl $\mathfrak{F} \subseteq TM$
integrable — closed under Lie bracket

foliation — mfld loc. looks like standard foliation of $\mathbb{R}^n$ by k-planes

Frobenius thm: distribution comes from foliation iff integrable

$\rightsquigarrow$ in char. 0, formal leaves exist for integrable distrib & are unique.

char $p>0$: integrability condition includes vanishing of p-curvature

$\rightsquigarrow$ given foliation $\mathfrak{F}$, formal leaves
- exist iff p-curvature vanishes
- are unique (up to constants of the relevant differential ring)

essentially, formal leaves / flat sections given by the Taylor expansion formula Katz 70 (5.1.7) see Vivien's talk

in particular: $\boxed{\text{p-curvature vanishes along formal flat sections}}$

p-curvature as obstruction to existence of flat sections, as curvature in char 0

essentially we'll need this for the pf.

Rem: weaker version:

vanishing of p-curvature to some order $\hat{=}$ flat section exists in some infinitesimal nbhd.

p-curvature for 1st order ODEs in dim 1

from lecture notes of Julien Roques

rk 1 on $\mathbb{A}^1$: $\nabla : \mathcal{O}_{\mathbb{A}^1} \to \Omega_{\mathbb{A}^1} : f \longmapsto df + f \cdot \omega$

triv. Ω^1 by dz for choice of coord.

eg. $\omega = b \cdot dz$ for some rat'l fn.,

flat section of ∇ corresp. to solution of $f' + bf = 0$

$$\left.\begin{aligned} (\nabla_{dz})^p(f) &= f^{(p)} + b_p \cdot f \\ \nabla_{(dz)^p}(f) &= f^{(p)} \end{aligned}\right\} \text{ using trivialization}$$

$\rightsquigarrow$ p-curvature $\psi_p = b_p$, def'd by $(\partial_z + b)^p = \partial_z^p + b_p$

inductive formula $b_{k+1} = b_k' + b \cdot b_k$, $b_1 = b$.

items in Katz' Taylor formula.

Jacobson formula: $\boxed{b_p = b^{(p-1)} + b^p}$

Similar for higher order, write as system of 1st order linear ODEs. (as usual)

Some random examples

expl: 1st order ODEs & their 3-curvature

① $y' + \frac{1}{x^2+1} y = 0$ $\qquad \psi_3 = \frac{1}{x^6+1}$

② $y' + \frac{x}{x^4+1} y = 0$ $\qquad \psi_3 = \frac{x^3}{x^{12}+1}$ vanishes to order 3 at 0.

expl: power series expansion for 1st order ODEs

① recursion for power series sol^n: $(n-1)c_{n-1} + (n+1)c_{n+1} - c_n = 0$

for $n=2$ forces $c_0 = 0$ (nonvanishing p-curv., no solution)

② recursion $(n-3)c_{n-3} + (n+1)c_{n+1} + c_{n-1} = 0$

for $n=5$ forces $c_0 = 0$ (get further because of vanishing of p-curv. to order 3)

Same happens for power series solution of Picard Fuchs equation, for $p \equiv 3 \bmod 4$, recursion $(n+1)(n+2)c_{n+2} = (2n+1)^2 c_n$ forces $c_0 = 0$

$\stackrel{?}{=}$ nonvanishing of p-curvature at $p \equiv 3 \bmod 4$

should be explained by Taylor formula in Cartier's thm.

§7 conjugate filtration

conjugate filtration recap (from Dongmin's talk)

$f: X \to S$ sm. proj / field k of char $p>0$

filtration $\tau_{\leq a}\Omega^\bullet_{X/S}$ of $\Omega^\bullet_{X/S}$ $\left\{\begin{array}{ll} 0 & \text{in deg} > a \\ \ker d & \text{in deg} = a \\ \Omega^i_{X/S} & \text{in deg } i < a. \end{array}\right.$

induces (decreasing) conjugate filtration

$$F^r_{conj} R^{s+r} f_* \Omega^\bullet_{X/S} = \operatorname{Im}\left(R^{s+r} f_* \tau_{\leq s}\Omega^\bullet_{X/S} \to R^{s+r} f_* \Omega^\bullet_{X/S}\right)$$

(abutment of hypercohom. spectral seq

$$E_2^{i,j} = R^i f_* \mathcal{H}^j(\Omega^\bullet_{X/S}) \Rightarrow R^{i+j} f_* \Omega^\bullet_{X/S} \quad)$$

results of Katz:

$$gr^a_{conj} R^{a+b} f_* \Omega^\bullet_{X/S} \simeq F^*_{abs}\, gr^b_{Hodge} R^{a+b} f_* \Omega^\bullet_{X/S}$$

Rem: over $\mathbb{C}$, cplx conj. $F_{abs}: \mathbb{C} \to \mathbb{C}$ acts on $H^n(Y, \Omega^\bullet_{Y/\mathbb{C}})$

$\rightsquigarrow$ iso $F^*_{abs} H^n(Y, \Omega^\bullet_{Y/\mathbb{C}}) \simeq H^n(Y, \Omega^\bullet_{Y/\mathbb{C}})$

$F^*_{abs}(F^i_{Hodge}) \longmapsto F^i_{conj}$ $\sim$ complex conjugate of Hodge filtration

conjugate filtration splits Hodge filtration, i.e. $H^n(Y, \Omega^\bullet_{Y/\mathbb{C}}) \simeq F^i_{Hodge} \oplus F^{n+1-i}_{conj}$

& get $H^{i,n-i} = F^i_{Hodge} \cap F^{n-i}_{conj}$

$$\rightsquigarrow gr^a_{conj} H^n(Y, \Omega^\bullet_{Y/\mathbb{C}}) \simeq F^{n-a}_{conj} \cap F^a_{conj} \simeq gr^{n-a}_{Hodge} H^n(Y, \Omega^\bullet_{Y/\mathbb{C}})$$

analog of char $p>0$ statement in Katz' theorem

initial conditions & conjugate filtration

Lemma (Cor. 4.2.2 in Corn.-Litt)

$R \subseteq \mathbb{C}$ f.g. $\mathbb{Z}$-alg, X/R sm. proj.

$0 \neq v \in H^1_{dR}(X/R)$

Then there exists a Zariski-dense set of closed pts $\mathfrak{p}_i$, $i \in I$ of $\operatorname{Spec} R$, s.th.

$$\boxed{v \bmod \mathfrak{p}_i \notin F^1_{conj} H^1_{dR}(X_{\mathfrak{p}_i}/\kappa(\mathfrak{p}_i))}$$

Pf sketch:

1. $H^1_{dR}(X/R) \simeq \operatorname{Ext}^1_{MIC}(\mathcal{O}_X, \mathcal{O}_X)$

modules w/ integrable conn.

$\rightsquigarrow$ represent $v \in H^1_{dR}$ as extension

$$0 \to (\mathcal{O}_X, d) \to (E_v, \nabla) \to (\mathcal{O}_X, d) \to 0.$$

2. $v \bmod \mathfrak{p}_i \in F^1_{conj} H^1_{dR}(X_{\mathfrak{p}_i}/\kappa(\mathfrak{p}_i))$

$\iff$ image of v in $H^0(\mathcal{H}^1_{dR}(X_{\mathfrak{p}_i}/\kappa(\mathfrak{p}_i)))$ is zero

i.e. v locally trivial

$\rightsquigarrow (E_v, \nabla)|_{X_{\mathfrak{p}_i}}$ loc. trivial (as well w/ flat connection)

3. results of André, Bost, Chudnovsky-Chudnovsky

(Grothendieck-Katz conj. for (E_v, ∇) of the above form)

$\Rightarrow (E_v, \nabla)$ trivial, i.e. $v = 0$ ∎

Rem: similar result for $\operatorname{Sym}^n(E, \nabla)$,

reduce to $n=1$ case via Veronese emb.

$$\mathbb{P} H^1_{dR} \to \mathbb{P} \operatorname{Sym}^n H^1_{dR}$$

Rem: Ogus Prop. VI. 2.6. in Deligne – Milne – Ogus – Shih
attributed to Katz

more direct argument for X ell. curve $/ R$ étale over $\mathbb{Z}$

① – $v \in F^1_{Hodge}$ $\rightsquigarrow$ for p s.th. X_p is ordinary

$F^1_{conj} \cap F^1_{Hodge} = 0$ in $H^1_{dR}(X_p/\kappa(p))$

+ ∞ many ordinary primes by result of Serre

$\rightsquigarrow$ ∞ many primes w/ $v \bmod p \notin F^1_{conj} H^1_{dR}(X_p/\kappa(p))$

② – general case: assume $v \bmod p \in F^1_{conj}$ for almost all p

for p supersingular $v \bmod p \in F^1_{Hodge}$

+ ∞ many supersingular primes by theorem of Elkies

apply special case

Rem: Does ordinary vs supersingular have an effect on char. p holonomy for ell. curves?