Oberseminar
Algebraicity of solutions to differential equations after Katz, Lam-Litt

SoSe 25

Introduction

The seminar is about the Grothedieck-Katz conjecture and related new conjectures by Lam and Litt and about the unconditional results concerning Gauss-Manin connections.

Let X be a smooth connected quasi-projective scheme over the complex numbers. The $\mathbb{C}$-rational points of X have the structure of a complex manifold denoted by X^{an}. A (holomorphic) connection on X^{an} is a locally free $\mathcal{O}_{X^{\mathrm{an}}}$-module E of finite rank together with a $\mathbb{C}$-linear map $\nabla : E \to E \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \Omega^1_{X^{\mathrm{an}}/\mathbb{C}}$ which satisfies the Leibniz rule $\nabla(ae) = e \otimes da + a\nabla(e)$, for local sections $e \in E$, $a \in \mathcal{O}_{X^{\mathrm{an}}}$. Such a connection gives rise to a continuous family of linear ordinary differential equations on X^{an}. Indeed if $U \subset X^{\mathrm{an}}$ is open such that $E_{|U} = \mathcal{O}_U^r$, then ∇ is given by an $r \times r$-matrix A of holomorphic 1-forms and the kernel of ∇ on U consists the vectors $(f_i) \in \mathcal{O}_U^r$ satisfying

$$(df_i) = A(f_i).$$

Up to shrinking U, the solution space of this differential equation is an r-dimensional $\mathbb{C}$-vector space. Analytic continuation of a solution along a loop centered at some base point defines a new solution on U. This process gives rise to the monodromy representation $\pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_r(\mathbb{C})$ defined by (E, ∇). Surprisingly any representation $\pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_r(\mathbb{C})$ (which only depends on the topological space X^{an}) arises in such a way and by a result of Deligne even more surprisingly (answering Hilbert's 21st Problem) one can even take (E, ∇) as the analytification of an algebraic connection $(E_{\mathrm{alg}}, \nabla_{\mathrm{alg}})$ on X which is regular singular at infinity and is uniquely determined. Since the differential equation defined by (E, ∇) has algebraic origin it is natural to ask when there is a full set of algebraic solutions as well, or equivalently when the corresponding monodromy representation has finite image. The Grothendieck-Katz conjecture says that this should be the case precisely when the reduction of $(E_{\mathrm{alg}}, \nabla_{\mathrm{alg}})$ modulo almost all primes p has vanishing p-curvature. Here the p-curvature of a connection is a notion which is particular for connections on schemes in positive characteristic.

In fact it suffices to prove the conjecture for $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In general the conjecture is far open. The best general result is Katz' Theorem, saying that the conjecture is true for Gauss-Manin connections. These are connections which can be canonically associated to a smooth morphism of $\mathbb{C}$-schemes $X \to S$.

There is some recent progress: Lam and Litt introduced a new in a sense more general conjecture, which also allows to incorporate non-linear ODE's, to decide whether a single formal solution can be algebraic, or if formal isomonodromic deformations of a given connection are algebraic. Lam and Litt prove these conjectures in the Gauss-Manin context.

The proofs combine Hodge-theoretic methods with methods from positive characteristic algebraic geometry and p-adic arithmetic geometry in an intriguing way. In this seminar we aim to understand the basic statements and the main methods and ideas entering in the proofs.

The Talks

1. (09.04.25) Introduction and discussion

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- (16.04.25) No talk

2. (23.04.25) Local Systems, Monodromy Representation and Connections

Let X be a connected complex manifold with a base point $x_0 \in X$. Define a *local system on X (with complex coefficients)* as a locally constant sheaf F of finite dimensional $\mathbb{C}$-vector spaces on X. Define the monodromy representation $T^F : \pi_1(X, x_0) \to \mathrm{Gl}_r(\mathbb{C})$ associated to F, where $r = \mathrm{rk}F$, following [Sab07, 15.d] relying on [Sab07, Lemma 15.5]. State and explain [Sab07, Theorem 15.8]. (Equivalence "local systems of rank r" with "r-dimensional complex representations of $\pi_1(X, x_0)$".)

Then define a (flat) connection on X, give the local description and basic operations, as in 11.1–11.3, 11.a, 11.b, 12.2, 12.4. of [Sab07]. Note that these notions can as well be defined algebraically on schemes over any base field. Explain the relation to ODE's as around (12.7) and note that this also includes higher order linear homogeneous equations as explained on the bottom of page 537 (first page of the article) of [Kat76]. Then state and explain [Del70, I, Thm 2.17] (see [Sab07, Thm 12.8] for a proof). Putting everything together we can associated to an ODE a monodromy representation. Explain how this works in case of $z\partial f/\partial z = \alpha f$ on $\mathbb{C} \setminus \{0\}$, where $\alpha \in \mathbb{C}$, see [Kat76, p. 539].

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3. (30.04.25) Regular Singular connections and the Riemann-Hilbert Correspondence

Go through [Kat76, pp. 554–551]. In particular explain GAGA, the notion of simple normal crossing divisor (= "a union of smooth divisors which cross transversally"), Deligne's definition of regular singular connection (= "algebraic differential equations with regular singular points at infinity"), and state the main results in the form [Del70, II, Thm 5.9] but follow the proof in [Kat76] in particular explain the Key Lemma on p. 547. As a corollary from what we saw in the last talk, say that we get an equivalence between the category of complex local systemes on the complex manifold defined by a smooth quasi-projective $\mathbb{C}$-scheme X and flat regular singular algebraic connections on X. This is the classical Riemann-Hilbert correspondence. See also [Del70, II, 5.] for details.

If time remains you could end by mentioning that (in characteristic zero) an integrable connection is an $\mathcal{O}$-coherent $\mathcal{D}$-module and that the Riemann-Hilbert correspondence extends to an equivalence between the derived category of bounded complexes of $\mathcal{D}$-modules with regular holonomic cohomology and the derived category of bounded complexes of constructible sheaves of $\mathbb{C}$-vector spaces, see, e.g., [Tey, 1.5]. But this is not needed in the following.

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4. (07.05.25) The Gauss-Manin connection

Explain the construction of the Gauss-Manin connection following [KO68, 2.] (see in particular Theorem 1). See also [Kat72, 1.4]. Show that it satisfies Griffiths' transversality, see [Kat72, Prop (1.4.1.6)]. State that the Gauss-Manin connection is regular-singular as in [Del70, Thm 7.9] (it suffices to do this for $\mathcal{V}$ the trivial connection). Then explain the Theorem of Brieskorn that the monodromy representation defined by a Gauss-Manin connection over a curve is quasi-unipotent at infinity following [Del70, III, 2.]. If time remains explain that certain hypergeometric differential equations "come from" Gauss-Manin connections of certain curves, see [Kat72, Prop (6.8.6) and around].

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5. (14.05.25) The Grothendieck-Katz conjecture

Define the p-curvature of a connection in characteristic $p > 0$ following [Kat70, (5.0)] and give Cartier's Theorem [Kat70, Theorem 5.1]. Just for sake of completeness state [Kat70, Prop 5.2]. Then state the Grothendieck-Katz conjecture in the form [Kat72, (I quat), p. 3] and discuss the other incarnations [Kat72, (I) – (I log), p. 1–2]. Explain that the general conjecture is equivalent to [Esn23, Conj 2.7], in particular it suffices to consider $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. See also [Lit, 4.2] and discuss [Lit, Example 4.2.4]. Then state [Kat72, Thm (5.1)] (the Grothendieck-Katz conjecture is true for Gauss-Manin connections). After some Hodge-theoretic preliminaries in the next two talks the proof will be given in talk 8. If time remains you can say a little bit on the state of the art of the conjecture, see the last paragraph in [Lit, 4.2, p. 30].

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6. (21.05.25) Excursion to Hodge-theory

This talk does not use the talks before and is about abstract Hodge theory and some of its consequences. Go through [Kat72, 4.2]. The aim is to understand the statement and the proof of [Kat72, Prop (4.2.2.3)]. In particular define the notions of *(a family of) pure/mixed Hodge structure(s)* and polarizations thereof. See also [Del71] and [PS08] for additional information on (mixed) Hodge structures.

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7. (28.05.25) Flatness of the Hodge filtration on the Gauss-Manin connection implies finite monodromy

Go through [Kat72, 4.3]. Explain the proofs of [Kat72, Prop (4.3.1) and Prop (4.3.3)] as detailed as possible. Proposition (4.3.3) is the first main ingredient in the proof of the Grothendieck-Katz conjecture for the Gauss-Manin connection.

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8. (04.06.25) Proof of Katz' Theorem

Define the Kodaira-Spencer class as in [Kat72, 1.1] and state (without proof) [Kat72, Prop (1.4.1.7)]. Then switch to positive characteristic and recall the definition of the inverse Cartier operator as in [Kat72, (2.1.1) Thm]. Then explain the statement of [Kat72, 3.2 Thm], which in view of the result above describes the Gauss-Manin connection on the Hodge graded pieces in terms of the p-curvature introduced in talk 5. This is the main technical result of [Kat72] and we use it as a black box. Proceed by recalling [Kat72, (5.1) Thm] and explain the proof. If time remains you can state [Kat72, 6.2 Thm] as an example application. The proof relies on the fact that the hypergeometric differential equation comes from a Gauss-Manin connection (as was maybe mentioned in talk 4).

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9. (18.06.25) The Conjecture of Lam and Litt

Introduce foliations, their leaves, the integrality of leaves etc. as in [LL, 2.1, 2.2], see also [Sab07, 13.b, 13.1 -13.3]. Explain how an integrable connection (E, ∇) gives rise to a foliation on $\mathbb{V}(E)$, see [Bos01, A.1]. Then state [LL, Conjecture 2.3.1] and explain how it implies [LL, Conj 1.1.1] and [LL, Conj 1.3.1]. If time remains say a word that [LL, Conj 6.1.1] would be a consequence of (a stacky version of Conj 2.3.1) as well as the Grothendieck-Katz Conjecture, see [LL, Prop 13.0.2].

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10. (25.06.25) Picard-Fuchs equations and cycle class initial conditions

Go through [LL, Sec 3], in particular state [LL, Thm 3.1.1] and explain the proof.

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11. (02.07.25) Families of elliptic curves

State [LL, Theorem 4.1.1] and explain its proof. Note that [LL, Theorem 4.2.1] already appeared in talk 8. (But you should still recall it.)

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12. (09.07.25) Main Theorem on non-linear differential equations

Go through [LL, Sec 6] and use whatever needed from section 7 to get clear statements. In particular define isomonodromic deformations of a connection (Def 7.1.1), state Conj 6.1.1, Def 6.1.2 and Theorem 6.1.3. Then try to explain as much of the proof as possible following the outline in 6.2.

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