Regular Singular Connections
and the Riemann–Hilbert Correspondence

by Julian Reichardt

Introduction

Let M be a connected complex manifold. In the last talk, we related the study of finite dimensional complex representations of the fundamental group $\pi_1(M)$ to the study of analytic differential equations on M. More precisely, we saw that every analytic differential equation on M gives rise to an associated complex representation of $\pi_1(M)$, its monodromy representation, and that this defines an equivalence of categories.

Now let X be a smooth connected complex variety and consider its associated analytification X^{an}, which will be a connected complex manifold. As a generalisation of the above, an algebraic differential equation on X now gives rise to an associated monodromy representation of $\pi_1(X^{\mathrm{an}})$. However, to obtain an equivalence of categories as before, we have to restrict the class of algebraic differential equations to these with regular singularities. This way, the equivalence also extends to the category of regular holonomic $\mathcal{D}_X$-modules on the one side and perverse sheaves on X on the other side.

Plan:

- recall analytification and Serre's GAGA theorems for (smooth) complex projective varieties
- for smooth complex projective varieties X, the construction of monodromy representations defines an equivalence of the category of flat connections on X and the category of finite dimensional complex representations of $\pi_1(X^{\mathrm{an}})$
- for smooth complex quasi-projective varieties X, the construction of monodromy representations defines an equivalence of the category of regular flat connections on X and the category of finite dimensional complex representations of $\pi_1(X^{\mathrm{an}})$
- mention the generalisation of the Riemann–Hilbert correspondence to regular holonomic $\mathcal{D}_X$-modules and perverse sheaves

1 Recollections: Analytification and Serre's GAGA

Denote by $(X, \mathcal{O}_X)$ an affine scheme of finite type over $\mathbb{C}$ and by $X^{\mathrm{an}} \stackrel{\text{def}}{=} X(\mathbb{C})$ its $\mathbb{C}$-rational points. By the Nullstellensatz, we can find complex *polynomial* functions $f_1, \ldots, f_r \colon \mathbb{C}^n \to \mathbb{C}$ such that X^{an} is given as the zero locus

$$X^{\mathrm{an}} = \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_r(x) = 0\}.$$

Subsets of this form are a special instance of so-called complex analytic sets, whose study leads to a type of locally ringed space generalising complex manifolds. Here we call a subset $A \subseteq \mathbb{C}^n$ **complex analytic** if for each $x \in A$ there exists an open neighbourhood $x \in U \subseteq \mathbb{C}^n$ and *holomorphic* functions $f_1, \ldots, f_r \colon U \to \mathbb{C}$ such that

$$U \cap A = \{x \in U \mid f_1(x) = \cdots = f_r(x) = 0\}.$$

Given an analytic set A and an open subset $U \subseteq A$, its **sheaf of holomorphic functions $\mathcal{H}_A$** has local sections

$$\mathcal{H}_A(U) \stackrel{\text{def}}{=} \{f \colon U \to \mathbb{C} \mid \forall z \in U\, \exists U_z \subseteq \mathbb{C}^n \text{ open neighbourhood of } z \text{ and } g \in \mathcal{H}_{\mathbb{C}^n}(U_z) : g|_{U_z \cap U} = f|_{U_z \cap U}\}$$

with $\mathcal{H}_{\mathbb{C}^n}$ the sheaf of holomorphic functions on $\mathbb{C}^n$. A **complex analytic space** is a locally ringed space admitting a cover by complex analytic sets.

The procedure constructing a complex analytic set from an affine scheme of finite type over $\mathbb{C}$ is compatible with gluing. This way, we can construct for any scheme X of locally finite type over $\mathbb{C}$ a complex analytic space, called its **analytification** X^{an}. The analytification construction moreover translates algebro-geometric notions to their correct complex-geometric counterparts. For instance, if X is a smooth complex (projective) variety, then X^{an} is a complex (projective) manifold.

There is a morphism of locally ringed spaces $\iota\colon (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$, which is universal among all morphisms of locally ringed spaces with domain a complex analytic space and codomain $(X, \mathcal{O}_X)$. Thus, for any (algebraic) $\mathcal{O}_X$-module $\mathcal{F}$ we can define an (analytic) $\mathcal{O}_{X^{\mathrm{an}}}$-module $\mathcal{F}^{\mathrm{an}}$ as the pullback

$$\mathcal{F}^{\mathrm{an}} \overset{\mathrm{def}}{=} \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{F}\,.$$

Analytification defines a functor from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_{X^{\mathrm{an}}}$-modules. If X is a *projective* variety, analytification defines an equivalence between the category of algebraic coherent $\mathcal{O}_X$-modules and analytic coherent $\mathcal{O}_{X^{\mathrm{an}}}$-modules, preserving cohomology.

Theorem 1 (GAGA). *Let X be a complex projective variety.*

(1) For every coherent algebraic sheaf $\mathcal{F}$ on X, and for every integer $q \geq 0$, the natural comparison morphism

$$\mathrm{H}^q(X, \mathcal{F}) \to \mathrm{H}^q(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

is an isomorphism.

(2) If $\mathcal{F}$ and $\mathcal{G}$ are two coherent algebraic sheaves on X, for every analytic morphism $\psi\colon \mathcal{F}^{\mathrm{an}} \to \mathcal{G}^{\mathrm{an}}$, there is a unique algebraic morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi^{\mathrm{an}} = \varphi$.

(3) For every coherent analytic sheaf $\mathcal{M}$ on X^{an}, there exists a coherent algebraic sheaf $\mathcal{F}$ on X such that $\mathcal{F}^{\mathrm{an}} \cong \mathcal{M}$ and $\mathcal{F}$ is unique up to isomorphism. □

Remark. The name *GAGA* derives as abbreviation of the name of Serre's original paper "Géométrie algébrique et géométrie analytique". Moreover, one can show that the analytification functor defines an equivalence $\mathsf{D}^{\mathrm{b}}_c(X) \to \mathsf{D}^{\mathrm{b}}_c(X^{\mathrm{an}})$ of derived categories of coherent sheaves. ⌟

Example. The analytification of the n-dimensional affine scheme $\mathbb{A}^n_{\mathbb{C}}$ is the n-dimensional complex space $\mathbb{C}^n$. Consequently, the analytification of the n-dimensional projective scheme $\mathbb{P}^n_{\mathbb{C}}$ is the n-dimensional complex projective space $\mathbb{CP}^n$. ⌟

Example. By definition, $(\mathcal{O}_X)^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{O}_X \cong \mathcal{O}_{X^{\mathrm{an}}}$. If X is smooth, then X^{an} is a complex manifold. Moreover, analytification defines an isomorphism $\left(\Omega^1_{X/\mathbb{C}}\right)^{\mathrm{an}} \cong \Omega^1_{X^{\mathrm{an}}/\mathbb{C}}$, where $\Omega^1_{X^{\mathrm{an}}/\mathbb{C}}$ is the $\mathcal{O}_{X^{\mathrm{an}}}$-module of holomorphic differential forms on the complex manifold X^{an}. ⌟

2 The Compact Problem and its Solution

We recall the notions of analytic differential equations on complex manifolds and algebraic differential equations smooth complex varieties, respectively. These are defined in terms of flat connections on locally free sheaves.

Definition 2. Let M be a connected complex manifold and X a smooth connected complex variety.

(1) An **analytic differential equation on M** is a pair $(\mathcal{M}, \nabla)$ of $\mathcal{M}$ a coherent locally free $\mathcal{O}_M$-module $\mathcal{M}$ and ∇ a flat connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_M} \Omega^1_{M/\mathbb{C}}$, with $\Omega^1_{M/\mathbb{C}}$ the holomorphic differential forms.

(1) An **algebraic differential equation on X** is a pair $(\mathcal{E}, \nabla)$ of $\mathcal{E}$ a coherent locally free $\mathcal{O}_X$-module $\mathcal{M}$ and ∇ a flat connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{C}}$, with $\Omega^1_{X/\mathbb{C}}$ the algebraic differential forms. ⌟

Remark. We will use "differential equation" and "flat connection" interchangeably. ⌟

An analytic differential equation $(\mathcal{M}, \nabla)$ on a connected complex manifold M gives rise to a finite dimensional complex representation of $\pi_1(M)$. Similarly, an algebraic differential equation $(\mathcal{E}, \nabla)$ on a smooth connected complex variety gives rise to a finite dimensional complex representation of $\pi_1(X^{\mathrm{an}})$. Representations of fundamental groups obtained this way are known as **monodromy representations** associated to differential equations. We are interested in the following question:

> Does the construction of monodromy representations define an equivalence of categories between *a* category of differential equations and the category of finite dimensional complex representations of the fundamental group?

Let M be a connected complex manifold. Last time, we established that *any* finite dimensional complex representation of $\pi_1(M)$ arises as the monodromy representation associated to an analytic differential equation on M. More precisely, we saw the following.

Theorem 3. *Let M be a connected complex manifold. The following categories are equivalent:*

- $\mathsf{Rep}_{\mathbb{C}}(\pi_1(M))$, *the category of finite dimensional complex representations of* $\pi_1(M)$*;*
- $\mathsf{Loc}_{\mathbb{C}}(M)$, *the category of local systems of complex vector spaces on* M*;*
- $\mathsf{Conn}(M)$, *the category of flat connections on* M*.* □

Now suppose that X is a smooth connected algebraic variety. If X is moreover assumed to be projective, we can use GAGA (Theorem 1) to identify the categories of flat algebraic connections on X and flat analytic connections on X^{an}.

Theorem 4. *Let X be a smooth connected projective variety. The following categories are equivalent:*

- $\mathsf{Conn}(X)$, *the category of algebraic flat connections on* X*;*
- $\mathsf{Conn}(X^{\mathrm{an}})$, *the category of analytic flat connections on* X^{an}*.* □

In combination, Theorem 3 for $M = X^{\mathrm{an}}$ and Theorem 4 show that for a smooth connected projective complex variety X, *any* finite dimensional complex representation of $\pi_1(X^{\mathrm{an}})$ arises as the monodromy representation associated to an algebraic differential equation on X.

Corollary 5. *Let X be a smooth connected projective variety. The following categories are equivalent:*

- $\mathsf{Rep}_{\mathbb{C}}(\pi_1(X^{\mathrm{an}}))$, *the category of finite dimensional complex representations of* $\pi_1(X^{\mathrm{an}})$*;*
- $\mathsf{Loc}_{\mathbb{C}}(X^{\mathrm{an}})$, *the category of local systems of complex vector spaces on* X^{an}*;*
- $\mathsf{Conn}(X)$, *the category of flat connections on* X*.* □

3 The Non-Compact Problem and its Solution

Now suppose that X is merely a smooth connected complex *quasi*-projective variety. Then the conclusion of Corollary 5 fails dramatically. Indeed, consider the case $X = \mathbb{A}^1$ (thus $X^{\mathrm{an}} = \mathbb{C}$) and consider the trivial 1-dimensional complex representation of $\pi_1(X^{\mathrm{an}}) = 0$. For any polynomial $p(z) \in \mathbb{C}[z]$, the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}z} = p(z) \cdot f$$

has solution

$$f = \exp\left(\int_0^z p(t)\,\mathrm{d}t\right).$$

This is an entire singlevalued function in z, hence has no monodromy.

This shows that infinitely many pairwise distinct algebraic differential equations on X give rise to the same monodromy representation of $\pi_1(X^{\mathrm{an}})$. To obtain an equivalence in the spirit of Corollary 5 we will restrict the type of algebraic differential equations we allow.

3.1 Reduction to the Key Lemma

The main idea underlying our approach of the quasi-projective case is to consider *suitable* compactifications to which we can then apply the GAGA theorems (Theorem 1). There are various subtleties to this strategy, which we will elaborate on now. Central to this approach is Hironaka's theorem on resolution of singularities, which we employ to produce these suitable compactifications.

Theorem 6 (Hironaka). *Let X be a smooth complex quasi-projective variety. Then there exsists a smooth projective complex variety $\tilde{X}$, containing X as a dense open subset, and such that the closed set $D = \tilde{X} \setminus X$ is a simple normal crossing divisor.* □

Remark. Given two such $\tilde{X}_1$ and $\tilde{X}_2$, there exists another $\tilde{X}_3$ "dominating" $\tilde{X}_1$ and $\tilde{X}_2$. ⌟

A **simple normal crossing divisor** is a union of smooth divisors which cross transversally. This means that (étale) locally at a point where r divisors $D_1, \dots, D_r$ cross, we may choose local coordinates $t_1, \dots, t_n$ such that D_i is defined by $t_i = 0$ for $i \leq r$.

Denote $\boldsymbol{\mathcal{D}er_D(\tilde{X}/\mathbb{C})}$ the subsheaf of $\mathcal{D}er(\tilde{X}/\mathbb{C})$ consisting of those derivations which preserve the ideals of each of the D_i. Let $\theta \in \mathcal{D}er_D(\tilde{X}/\mathbb{C})$ be an (étale) local section and write $\theta = \sum_{i=1}^n f_i \partial_{t_i}$, using that $\mathcal{D}er(\tilde{X}/\mathbb{C})$ is (étale) locally free with basis $\partial_{t_1}, \dots, \partial_{t_n}$. Since θ preserves the defining ideals of the divisors D_i, we conclude that $\theta(t_i) = f_i \in (t_i)$ for all $i \leq r$. Thus, $\mathcal{D}er_D(\tilde{X}/\mathbb{C})$ is (étale) locally free with basis $t_1\partial_{t_1}, \dots, t_r\partial_{t_r}, \partial_{t_{r+1}}, \dots, \partial_{t_n}$. Recall that there is a natural duality pairing

$$\mathcal{D}er(\tilde{X}/\mathbb{C}) \otimes \Omega^1_{\tilde{X}/\mathbb{C}} \to \mathcal{O}_{\tilde{X}}$$

which identifies $\Omega^1_{\tilde{X}/\mathbb{C}}$ with the linear dual of $\mathcal{D}er(\tilde{X}/\mathbb{C})$. This pairing is linearly extended from the local equality $\mathrm{d}t_i(\partial_{t_j}) = \delta_{ij}$. We define the $\mathcal{O}_{\tilde{X}}$-linear dual of $\mathcal{D}er_D(\tilde{X}/\mathbb{C})$ to be $\boldsymbol{\Omega^1_{\tilde{X}/\mathbb{C}}(\log D)}$. Consequently, the dual of $t_i\partial_{t_i}$ is $\frac{\mathrm{d}t_i}{t_i}$. Thus, $\Omega^1_{\tilde{X}/\mathbb{C}}(\log D)$ is (étale) locally free with basis $\frac{\mathrm{d}t_1}{t_1}, \dots, \frac{\mathrm{d}t_r}{t_r}, \mathrm{d}t_{r+1}, \dots, \mathrm{d}t_n$, justifying the notation.

Definition 7. Let X be a smooth complex quasi-projective variety. Let $(\mathcal{E}, \nabla)$ be an algebraic flat connection on X and let $X \to \tilde{X}$ be a compactification as in Theorem 6 with $D = \tilde{X} \setminus X$. Then $(\mathcal{E}, \nabla)$ is **regular** if there is a locally free $\mathcal{O}_{\tilde{X}}$-module $\overline{\mathcal{E}}$ extending $\mathcal{E}$ together with a morphism $\overline{\nabla} \colon \overline{\mathcal{E}} \to \overline{\mathcal{E}} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}/\mathbb{C}}(\log D)$ satisfying the Leibniz rule and extending ∇. ⌟

Remark. This definition is independent of both the chosen compatification $\tilde{X}$ and the extension $(\overline{\mathcal{E}}, \overline{\nabla})$. ⌟

We are now ready to state the *Key Lemma*.

Lemma 8. *Let X be a smooth complex quasi-projective variety. Let $(\mathcal{M}, \nabla)$ be an analytic differential equation on X^{an} and let $X \to \tilde{X}$ be a compactification as in Theorem 6 with $D = \tilde{X} \setminus X$. Then there exists a unique locally free $\mathcal{O}_{\tilde{X}^{\mathrm{an}}}$-module $\overline{\mathcal{M}}$ on $\tilde{X}^{\mathrm{an}}$ extending $\mathcal{M}$ together with a morphism $\overline{\nabla} \colon \overline{\mathcal{M}} \to \overline{\mathcal{M}} \otimes_{\mathcal{O}_{\tilde{X}^{\mathrm{an}}}} \left(\Omega^1_{\tilde{X}/\mathbb{C}}(\log D)\right)^{\mathrm{an}}$ satisfying the Leibniz rule and extending ∇.*

Theorem 9. *Let X be a smooth connected quasi-projective variety. The following categories are equivalent:*

- $\mathsf{Conn}_r(X)$*, the category of regular flat connections on X;*
- $\mathsf{Conn}(X)$*, the category of flat connections on X^{an}.*

Proof. Every algebraic (regular) flat connection $(\mathcal{E}, \nabla)$ on X gives rise to analytic flat connection $(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ on X^{an} by analytification.

Conversely, let $(\mathcal{M}, \nabla)$ be an analytic flat connection on X^{an}. By the Key Lemma (Lemma 8), there exists a locally free $\mathcal{O}_{\tilde{X}^{\mathrm{an}}}$-module $\overline{\mathcal{M}}$ on $\tilde{X}^{\mathrm{an}}$ extending $\mathcal{M}$ and a morphism $\overline{\nabla} \colon \overline{\mathcal{M}} \to \overline{\mathcal{E}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\tilde{X}^{\mathrm{an}}}} \left(\Omega^1_{\tilde{X}/\mathbb{C}}(\log D)\right)^{\mathrm{an}}$ extending ∇.

Since $\tilde{X}$ is a complex projective variety, we case use the GAGA equivalence (Theorem 1) to produce a locally free $\mathcal{O}_{\tilde{X}}$-module $\overline{\mathcal{E}}$, whose analytification is $\overline{\mathcal{M}}$, and a morphism $\overline{\nabla} \colon \overline{\mathcal{E}} \to \mathcal{E} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega^1_{\tilde{X}/\mathbb{C}}(\log D)$, whose analytification is $\overline{\nabla}$. By definition, the restriction $(\mathcal{E}, \nabla) \stackrel{\text{def}}{=} (\overline{\mathcal{E}}|_X, \overline{\nabla}|_X)$ is then a algebraic regular flat connection.

Consider the analytification $(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$. Observe that $(\overline{\mathcal{M}}, \overline{\nabla}) = (\overline{\mathcal{E}}, \overline{\nabla})^{\mathrm{an}}$ is an extension of $(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}})$ to $\tilde{X}^{\mathrm{an}}$ in the sense of Lemma 8. By the uniqueness assertion of Lemma 8, this implies that $(\overline{\mathcal{E}^{\mathrm{an}}}, \overline{\nabla^{\mathrm{an}}}) = (\overline{\mathcal{M}}, \overline{\nabla})$, hence $(\mathcal{E}^{\mathrm{an}}, \nabla^{\mathrm{an}}) = (\overline{\mathcal{E}^{\mathrm{an}}}|_X, \overline{\nabla^{\mathrm{an}}}|_X) = (\overline{\mathcal{M}}|_X, \overline{\nabla}|_X) = (\mathcal{M}, \nabla)$ as needed. □

Corollary 10 (Riemann–Hilbert Correspondence). *Let X be a smooth connected quasi-projective variety. The following categories are equivalent:*

- $\mathsf{Rep}_{\mathbb{C}}(\pi_1(X^{\mathrm{an}}))$, *the category of finite dimensional complex representations of* $\pi_1(X^{\mathrm{an}})$*;*
- $\mathsf{Loc}_{\mathbb{C}}(X^{\mathrm{an}})$, *the category of local systems of complex vector spaces on* X^{an}*;*
- $\mathsf{Conn}_r(X)$, *the category of regular flat connections on* X. □

3.2 On the Proof of the Key Lemma

It remains to prove Lemma 8. The proof strategy can be summarised as consisting of the following three steps:

- **Step 1:** Extend the analytic differential equation on X^{an} locally along D.
- **Step 2:** Show that our construction characterises the local extensions from **Step 1** uniquely.
- **Step 3:** Use the unique local characterisation from **Step 2** to globalise.

Step 1

Choose local coordinates $z_1, \dots, z_n$ of $\tilde{X}$ around a point where r divisors $D_1, \dots, D_r$ cross in such a way that the D_i are defined by the equations $z_i = 0$ for $i \leq r$. Consider a polydisk V of radius ε. Then $V \cap X^{\mathrm{an}}$ becomes the product of r punctured disks $0 < |z_i| < \varepsilon$, $1 \leq i \leq r$, and $n - r$ disks $|z_j| < \varepsilon$, $r + 1 \leq j \leq n$. Restricting $(\mathcal{M}, \nabla)$ to $X^{\mathrm{an}} \cap V$ produces an analytic differential equation, hence by Theorem 3 corresponds to a representation ρ of $\pi_1(V \cap X^{\mathrm{an}})$ on a finite dimensional $\mathbb{C}$-vector space L.

Since $V \cap X^{\mathrm{an}}$ is the product of r punctured disks and $n - r$ disks, its fundamental group is free abelian of rank r. Choose generators $\gamma_1, \dots, \gamma_r$ corresponding to the r punctured-polydisk-factors. The representation ρ is then completely determined by the choice of the r commuting automorphisms $\rho(\gamma_i)$ of L. In fact, one can choose *unique* endomorphisms B_j of L such that

1) $\exp(2\pi i B_i) = \rho(\gamma_i)$,
2) the eigenvalues of B_i have real parts in the strip $(-1, 0]$, and
3) the B_i mutually commute.

We now construct an extension of $(\mathcal{M}, \nabla)|_{\cap V \cap X^{\mathrm{an}}}$ to V. For this, let $\overline{\mathcal{M}} \overset{\mathrm{def}}{=} L \otimes \mathcal{O}_V$ and define

$$\begin{aligned} \overline{\nabla} \colon L \otimes \mathcal{O}_V &\to L \otimes \Omega^1_V(\log D) \\ \ell \otimes f &\mapsto f\left(-\sum_{i=1}^{r}(B_i \ell) \otimes \frac{\mathrm{d}z_i}{z_i}\right) + \ell \otimes \mathrm{d}f\,. \end{aligned}$$

This defines a local extension $(\overline{\mathcal{M}}, \overline{\nabla})$ on V of $(\mathcal{M}|_{V \cap X^{\mathrm{an}}}, \nabla|_{V \cap X^{\mathrm{an}}})$. Indeed, the restriction of $(\overline{\mathcal{M}}, \overline{\nabla})$ to $V \cap X^{\mathrm{an}}$ yields the correct monodromy representation by construction, hence has to isomorphic to $(\mathcal{M}|_{V \cap X^{\mathrm{an}}}, \nabla|_{V \cap X^{\mathrm{an}}})$ by Theorem 3. Here a fundamental solution matrix is given by

$$\prod_{i=1}^{r} z_i^{B_i} \overset{\mathrm{def}}{=} \exp\left(\sum_{i=1}^{r} B_i \log z_i\right).$$

Step 2

We claim that the extension from **Step 1** is uniquely characterised by the property that in any basis for the locally free $\mathcal{O}_V$-module, a fundamental solution matrix is of the form

$$H(z) \prod_{i=1}^{r} z_i^{B_i}$$

with $H(z) \in \mathrm{GL}(n, \mathcal{O}_V)$ and the B_i commuting matrices whose eigenvalues have real parts in the strip $(-1, 0]$.

To argue that the above condition characterises our extension, note that any other extension of $\mathcal{M}$ has to be of the form $A(z)\overline{\mathcal{M}}$ with $A(z)$ holomorphic on $X^{\mathrm{an}} \cap V$ but possibly with singularities along the D_i. The class of the extension $A(z)\overline{M}$ depends on the class of $A(z)$ in the double coset

$$\mathrm{GL}(n, \mathcal{O}_V) \diagdown \mathrm{GL}(n, \mathcal{O}_{V \cap X^{\mathrm{an}}}) \diagup \mathrm{GL}(n, \mathcal{O}_V)\,.$$

Suppose that the extension $A(z)\overline{M}$ has fundamental solution matrix given by

$$K(z) \prod_{i=1}^{r} z_i^{C_i}$$

with $K(z) \in \mathrm{GL}(n, \mathcal{O}_V)$ and with C_i commuting matrices whose eigenvalues have real parts in the strip $(-1, 0]$. This implies that

$$K(z) \prod_{i=1}^{r} z_i^{C_i} = A(z)^{-1} H(z) \prod_{i=1}^{r} z_i^{B_i}\,.$$

We conclude $\exp(2\pi C_i) = \exp(2\pi B_i)$ from which we deduce $C_i = B_i$ since the real parts of their eigenvalues lie in the strip $(-1, 0]$. In particular, we conclude that $K(z) = A(z)^{-1} H(z)$, which implies $A(z) \in \mathrm{GL}(n, \mathcal{O}_V)$. Thus, $A(z)\overline{M}$ and $\overline{\mathcal{M}}$ are isomorphic.

Step 3

Let us now argue that the local extension constructed in **Step 1** are locally compatible, using **Step 2**. Consider a polydisk V around an intersection point of $D_1, \ldots, D_r$ as in **Step 1**. Suppose that $V' \subset V$ is a polydisk around a point where only the divisors $D_1, \ldots, D_s$ for $s < r$ cross. Choose local coordinates $z_1, \ldots, z_n$ for V and $w_1, \ldots, w_n$ for V'.

For the local coordinates $z_1, \ldots, z_n$ on V, the fundamental solution matrix was

$$\prod_{i=1}^{r} z_i^{B_i}\,.$$

For $i \leq s$, both $z_i = 0$ and $w_i = 0$ define D_i near the intersection point of $D_1, \ldots, D_s$. Thus, there are invertible functions u_i such that $z_i = u_i w_i$. As the functions $z_{s+1}, \ldots, z_n$ do not pass through the intersection point, they are invertible themselves. Thus, by possibly shriking V', we can choose functions $v_1, \ldots, v_n$ such that

$$u_i = \exp(v_i) \qquad \text{and} \qquad z_j = \exp(v_j)$$

for $i = 1, \ldots, s$ and $j = s+1, \ldots, n$. Then we compute

$$\prod_{i=1}^{r} z_i^{B_i} = \prod_{i=1}^{s} (u_i w_i)^{B_i} \prod_{i=s+1}^{r} z_i^{B_i} = \exp\left(\sum_{i=1}^{r} B_i v_i\right) \prod_{i=1}^{s} w_i^{B_i}$$

and $\exp\left(\sum_{i=1}^{r} B_i v_i\right) \in \mathrm{GL}(n, \mathcal{O}_{V'})$ by construction. This shows, that our extensions are locally compatible and thereby concludes the proof of Lemma 8.

4 Complements on $\mathcal{D}_X$-modules

Let X be a smooth connected complex variety. The formalism of flat connections is robust enough to allow for more general $\mathcal{O}_X$-modules than locally free ones. This leads to the notion of a $\mathcal{D}_X$-*module*.

Definition 11. Let X be a smooth complex variety. A **(left) $\mathcal{D}_X$-module** is an $\mathcal{O}_X$-module $\mathcal{F}$ together with a $\mathbb{C}$-linear morphism

$$\nabla\colon \mathcal{D}er(X/\mathbb{C}) \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{F}), \qquad \theta \mapsto \nabla_\theta$$

satisfying

(1) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$ for all $f \in \mathcal{O}_X$, $\theta \in \mathcal{D}er(X/\mathbb{C})$, $s \in \mathcal{F}$;

(2) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$ for all $f \in \mathcal{O}_X$, $\theta \in \mathcal{D}er(X/\mathbb{C})$, $s \in \mathcal{F}$;

(3) $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$ for all $\theta_1, \theta_2 \in \mathcal{D}er(X/\mathbb{C})$, $s \in \mathcal{F}$. ⌟

Example. Let $(\mathcal{E}, \nabla)$ be a flat connection X. Recall that there is a $\mathbb{C}$-bilinear pairing

$$\mathcal{D}er(X/\mathbb{C}) \otimes \Omega^1_{X/\mathbb{C}} \to \mathcal{O}_X$$

of locally free $\mathcal{O}_X$-modules. The composition

$$\mathcal{D}er(X/\mathbb{C}) \otimes \mathcal{E} \to \mathcal{D}er(X/\mathbb{C}) \otimes (\Omega^1_{X/\mathbb{C}} \otimes_{\mathcal{O}_X} \mathcal{E}) \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$$

then defines the underlying $\mathcal{D}er(X/\mathbb{C})$-action for a $\mathcal{D}_X$-module structure on $\mathcal{E}$. Conversely, any $\mathcal{D}_X$-module which is coherent as $\mathcal{O}_X$-module comes from a flat connection. ⌟

In particular, an $\mathcal{D}_X$-module whose underlying $\mathcal{O}_X$-module is coherent leads back to the notion of flat connections. However, to get control over the class of $\mathcal{D}_X$-modules, we need to introduce *some* finiteness conditions. The natural choice of requiring coherence over $\mathcal{D}_X$-module is ill-behaved. One further restricts to so-called *holonomic* $\mathcal{D}_X$-modules, which assemble into a well-behaved category, stable under geometric operations. Moreover, one can extend the notion of *regularity* from flat connections on X to $\mathcal{D}_X$-modules on X in such a way, that regular flat connections are an example of regular (holonomic) $\mathcal{D}_X$-modules.

Recall that part of the assertion of Corollary 10 was an equivalence between the category of regular flat connections on X and the category of local systems of $\mathbb{C}$-vector spaces on X^{an}. If one extends the notion of regular flat connections on X to that of regular holonomic $\mathcal{D}_X$-modules on X, one can ask whether there is a corresponding notion of sheaves on X^{an} generalising local systems of $\mathbb{C}$-vector spaces accordingly.

This leads to so-called *perverse sheaves*. Perverse sheaves are (bounded) *complexes* of sheaves of $\mathbb{C}_{X^{\mathrm{an}}}$-modules satisfying certain cohomological restrictions. In particular, every local system on X^{an} gives rise to a perverse sheaf on X and Corollary 10 generalises as follows.

Corollary 12 (Riemann–Hilbert Correspondence). *Let X be a smooth complex variety. The following categories are equivalent:*

- $\mathsf{Reg}(X)$*, the category of regular holonomic $\mathcal{D}_X$-modules on X;*
- $\mathsf{Perv}(X)$*, the category of perverse sheaves on X.* □

More precisely, the equivalence of Corollary 12 is induced by an equivalence of derived categories. On the one side, we consider the derived category of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology and on the other side the derived category of complexes of $\underline{\mathbb{C}}_{X^{\mathrm{an}}}$-modules with algebraically constructible cohomology. The category of perverse sheaves is actually *defined* using this equivalence.

Theorem 13 (Riemann–Hilbert Correspondence). *Let X be a smooth complex variety. The following categories are equivalent:*

- $\mathsf{D}^b_{rg}(X)$*, the derived category of bounded complexes of $\mathcal{D}_X$-modules on X with regular holonomic cohomology;*
- $\mathsf{D}^b_c(X)$*, the category of bounded complexes of $\underline{\mathbb{C}}_{X^{\mathrm{an}}}$-modules with algebraically constructible cohomology.* □

To summarise, for a smooth connected complex variety X, Corollaries 10 and 12 and Theorem 13 assemble into a commuting diagram of inclusions and equivalences of categories:

$$\begin{array}{ccc}
\mathsf{Conn}_r(X) & \xleftrightarrow{\text{Corollary 10}} & \mathsf{Loc}_{\mathbb{C}}(X^{\mathrm{an}}) \\
\big\downarrow & & \big\downarrow \\
\mathsf{Reg}(X) & \xleftrightarrow{\text{Corollary 12}} & \mathsf{Perv}(X) \\
\big\downarrow & & \big\downarrow \\
\mathsf{D}^b_{rg}(X) & \xleftrightarrow{\text{Theorem 13}} & \mathsf{D}^b_c(X)
\end{array}$$