Hodge structures

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This talk is based on [Kat72, §4.2].

1 Definitions

Convention. A filtered object in an abelian category $\mathfrak{C}$ is an object X together with a sequence of subobjects $\underline{F} = \left(\cdots \subseteq F^1 \subseteq F^0 \subseteq F^{-1} \subseteq \cdots\right)$. If there are m, n such that $F^m = X, F^n = 0$, it's called finite.[1]

An ascendingly filtered object in an abelian category $\mathfrak{C}$ is an object X together with a sequence of subobjects $\overline{W} = (\cdots \subseteq W_{-1} \subseteq W_0 \subseteq W_1 \cdots)$. If there are m, n such that $W_m = 0, W_n = X$, it's called finite.

Definition 1.1. For a complex subspace $U \subset V$ together with an $\mathbb{R}$-skeleton $V = W \otimes_{\mathbb{R}} \mathbb{C}$ with $W \in \mathbb{R} - \underline{\mathfrak{Mod}}$ as an $\mathbb{R}$-vector space, let $\overline{U}$ be the complex conjugate subspace of U with respect to that basis. Putting $V (= W \otimes_{\mathbb{R}} \langle 1 \rangle) \oplus (W \otimes_{\mathbb{R}} \langle \mathrm{i} \rangle)$, conjugation acts by mapping $v \otimes \mathrm{i}$ to $-v \otimes \mathrm{i}$.

Definition 1.2. i) A (pure) $\mathbb{Z}$-Hodge structure H of weight n is a finitely generated $\mathbb{Z}$-module $H_{\mathbb{Z}}$ together with a finite filtration $\underline{F}$ of

$$H_{\mathbb{C}} \coloneqq H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \in \mathbb{C} - \underline{\mathfrak{Mod}}$$

such that

$$H_{\mathbb{C}} = \bigoplus_{p+q=n} \underbrace{F^p \cap \overline{F}^q}_{=: H^{p,q}}.$$

Put $H_{\mathbb{R}} \coloneqq H_{\mathbb{Z}} \otimes \mathbb{R}$. (Note that conjugation acts on the second factor in the tensor product.)

ii) Define an $\mathbb{R}$-linear group action of $\mathbb{C}^\times$ on $H_{\mathbb{C}}$ by $\mathrm{s}(z, v) = z^p \overline{z^q} v, v \in H^{p,q}, p + q = n$. One puts $\mathrm{C}(-) \coloneqq \mathrm{s}(\mathrm{i}, -)$.

iii) A morphism of pure Hodge-structures is a map $H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ compatible with s.

iv) The rank of a Hodge structure is the complex dimension of $H_{\mathbb{C}}$.

[1] The convention bit here is that a filtration is by standard descending.

v) Define an R-Hodge structure for a ring inclusion $\mathbb{Z} \subset R \subset \mathbb{R}$ by exchanging $\mathbb{Z}$-modules by R-modules, and $\otimes_{\mathbb{Z}}$ by $\otimes_R$. In particular, there is the notion of an $\mathbb{R}$-Hodge structure.

Between Hodge structures of different weights, there is only the zero morphism, due to $\mathrm{s}(2, f(u)) = 2^n f(u) = f(\mathrm{s}(2,u)) = 2^m f(u)$.[2]

Lemma 1.3 ([Del71, §2.1.11]). *Hodge structures form an abelian category equipped with the obvious internal hom and* $\otimes$. *The tensor product is defined by* $(H' \otimes H'')_{\mathbb{Z}} = H'_{\mathbb{Z}} \otimes H''_{\mathbb{Z}}$; *the filtration* $\underline{F}$ *is the graded* $\mathbb{C}$*-tensor product* $F^n = \bigoplus_{p+q=n} F'^p \otimes_{\mathbb{C}} F''^q$ *of the respective filtrations* $\underline{F}', \underline{F}''$ *where* n *is the sum of the weights of* H' *and* H''; *and so is the action* s.

The internal hom of two Hodge structures H, H' *of the same weight*[3]

$$(\underline{\mathrm{Hom}}(H,H'))_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(H,H'), \mathrm{s}(z,f)(x) = \mathrm{s}(z, f(\mathrm{s}(z^{-1},x))), f \in \underline{\mathrm{Hom}}(H',H'')_{\mathbb{C}}, x \in H''_{\mathbb{C}}.$$

The filtration $\underline{F}$ *of* $\mathrm{Hom}(H',H'')_{\mathbb{C}}$ *is defined by*

$$F^n = \left\{ f \in \mathrm{Hom}_{\mathbb{C}}(H'_{\mathbb{C}}, H''_{\mathbb{C}}) : f[F^i] \subset F^{i+n} \forall i \right\}.$$

(There are also dual spaces and outer products)

Example 1.4. The Tate Hodge structure $\mathbb{Z}(n)$ is the Hodge structure of rank 1 and weight $-2n$, which is given by

$$H_{\mathbb{Z}} = \mathbb{Z}, F^{-n} = H_{\mathbb{Z}}, F^{n+1} = 0.$$

We have $\mathbb{Z}(n) \cong \mathbb{Z}(1)^{\otimes n}$, which can be seen by counting degrees in the graded tensor product of the filtrations.

$\mathbb{Z}(n)_{\mathbb{Z}}$ is often understood as the subgroup $(2\pi \mathrm{i})^n \mathbb{Z} \subset \mathbb{C}$, which comes from Cauchy's integral theorem. (I will hopefully return to that later.)

Definition 1.5. A polarization of a weight n Hodge structure is a homomorphism

$$H \otimes H \xrightarrow{(\ ,\ )} \mathbb{Z}(-n)$$

such that the real bilinear form on $H_{\mathbb{R}}$ given by $(x,y) \mapsto (x, \mathrm{C}y)$ is symmetric positive definite. An object which admits a polarization (there are no uniqueness restrictions) is called polarizable.[4]

Definition 1.6. Define the homomorphisms $H \to H'$ up to isogeny as $\mathrm{Hom}(H,H') \otimes_{\mathbb{Z}} \mathbb{Q}$. This is equivalent to the morphisms of the category of Hodge structures localized at morphisms which have finite kernel and cokernel on the integral lattices.

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[2] Any number which is no root of unity could have been chosen instead of 2.

[3] If H', H'' are of weight n, m, there is a weight $m-n$ Hodge structure $H := \underline{\mathrm{Hom}}(H',H'')$ defined by $H_{\mathbb{Z}} := \left\{ f \in \mathrm{Hom}_{\mathbb{Z}}(H'_{\mathbb{Z}}, H''_{\mathbb{Z}}) : f[F'^{n+k}] \subset F''^{m+k+j} \right\}$ and $F^k := \left\{ f \in \mathrm{Hom}_{\mathbb{C}}(H'_{\mathbb{C}}, H''_{\mathbb{C}}) : f[F'^{n+j}] \subset F''^{m+k+j} \forall j \right\}$. The morphism group of Hodge structures $H' \to H''$ is zero nevertheless.

[4] In the literature, there is often a factor $(2\pi \mathrm{i})^n$ in front of $(x, \mathrm{C}y)$. This comes from the identification of $(2\pi \mathrm{i})^{-n}$ and $\mathbb{Z}(-n)$.

Remark 1.7. [5] Polarizations factor through isogeny. The polarizable Hodge structures up to isogeny form a full semi-simple subcategory, i. e. every object is a direct sum of simple objects.[6] Hence it is closed under internal hom, $\otimes$, finite $\oplus$ of same weight, subobjects and quotient objects H'/H'' of polarizable Hodge structures H', H''.

Definition 1.8. We form the category $\mathfrak{D}$ of polarized Hodge structures and the induced category $\mathfrak{D}'$ of polarized Hodge structures up to isogeny. A morphism up to isogeny of polarized Hodge structures $(H', (\ ,\ )), (H'', (\ ,\ ))$ is an element $f \in \mathrm{Hom}(H', H'') \otimes \mathbb{Q}$ such that under $\mathrm{Hom}(H', H'') \otimes \mathbb{Q} \hookrightarrow \mathrm{Hom}_{\mathbb{R}-\underline{\mathfrak{Mod}}}(H'_{\mathbb{R}}, H''_{\mathbb{R}})$, the image $H'_{\mathbb{R}} \xrightarrow{f} H''_{\mathbb{R}}$ commutes with $(\ ,\ )$.

Definition 1.9. A family of weight n pure Hodge structures on $\mathcal{S} \in \underline{\mathfrak{Top}}$ is a local system $H_{\mathbb{Z}}$ on $\mathcal{S}$ of $\mathbb{Z}$-modules of finite type, together with a continuously varying filtration $(\underline{F})_s$ of $(H_{\mathbb{C}})_s$ on the stalks, which is a pure Hodge structure on each stalk together with the Hodge filtration $(\underline{F})_s$. (A continuously/smoothly/... varying filtration means that it comes from a map $\mathcal{S} \to \mathbb{X}$ satisfying the desired property where $\mathbb{X}$ is a flag space of $H_{\mathbb{C}}$.) Morphisms are given by morphisms of local systems which are stalkwise Hodge structure morphisms.

A polarization is then a local system morphism $H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \xrightarrow{(\ ,\ )} \mathbb{Z}(-n)_{\mathcal{S}}$, where $\mathbb{Z}(-n)_{\mathcal{S}}$ is a constant sheaf together with the constant Hodge filtration, such that the pure Hodge structures on the stalks are $\mathbb{Z}(-n)$.

Note that the continuously varying filtration does NOT need to come from a filtration in the category of $\mathbb{C}$-local systems.

Remark 1.10. Remark 1.7 applies to families of Hodge structures with little change.

Definition 1.11.

i) A mixed Hodge structure is an ascendingly and finitely filtered object $(H_{\mathbb{Z}}, \overline{W}), \overline{W} = (0 \subseteq W_a \subseteq \cdots \subseteq W_b = H_{\mathbb{Z}})$, in $\mathbb{Z} - \underline{\mathfrak{Mod}}_{\text{fin. gen.}}$ together with a filtration $\underline{F}$ of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\mathrm{gr}_n^W := W_n H_{\mathbb{Z}} / W_{n-1} H_{\mathbb{Z}}$ is a pure Hodge structure of weight n for each n. The finite ascending filtration $\overline{W}$ is called weight filtration.

ii) A family of mixed Hodge structures on $\mathcal{S}$ is an ascendingly and finitely filtered object in the category of local systems of finitely generated $\mathbb{Z}$-modules $(H_{\mathbb{Z}}, \overline{W})$ together with a continuously varying Hodge filtration F_s^i of the stalks $(H_{\mathbb{C}})_s$ such that $\mathrm{gr}_n^W := W_n H_{\mathbb{Z}} / W_{n-1} H_{\mathbb{Z}}$ is a family of pure Hodge structures of weight n. The ascending filtration $\overline{W}$ is called weight filtration.

iii) The morphisms of families of mixed Hodge structures are local system morphisms compatible with both filtrations.

Lemma 1.12. *A morphism of mixed Hodge structures $(H', \overline{W}) \to (H'', \overline{W}')$ induces morphisms of pure Hodge structures* $\mathrm{gr}_n^{W'} \to \mathrm{gr}_n^{W''}$.

[5] For the case of fixed polarizations, see also [PS08, Corollary 2.12].

[6] A simple object is an object without any nontrivial subobjects.

Proof. Let $f\colon (H', \overline{W}') \to (H'', \overline{W}'')$ be a morphism. Since f respects $\overline{W}$, the maps on the graded pieces $\mathrm{gr}_n^{W'} = W'_n / W'_{n-1}$ are well-defined. f respecting $\underline{F}$ implies that $\left(\mathrm{gr}_n^{W'}\right)^{p,q} = F'^p \,\mathrm{gr}_n^{W'} \cap \overline{F'^q \,\mathrm{gr}_n^{W'}}$, where there's $p + q = n$, is mapped to $\left(\mathrm{gr}_n^{W''}\right)^{p,q}$. It follows from the definition of s that s commutes with f. □

There is a variation where W filters the tensor product $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, cf. [Del71, §2.3]. If one generalizes to R-Hodge structures for an arbitrary ring R, W may filter $H_R \otimes_{\mathbb{Z}} \mathbb{Q}$, cf. [PS08, Def. 3.1].

Fact 1.13. *For a morphism of families of mixed Hodge structures $\varphi\colon (H_{\mathbb{Z}}, \overline{W}) \to (H'_{\mathbb{Z}}, \overline{W}')$, there is $W'_i \cap \operatorname{Im} \varphi = \varphi[W_i]$.*

One calls this strict compatibility.

2 Propositions

Proposition 2.1. *For $\tilde{H} \coloneqq (H(\ ,\ ))$ a polarized Hodge structure, $\mathbf{Aut}_{\mathfrak{D}}(\tilde{H})$ is finite.*

Proof. Informally speaking, we can put together $\mathbf{Aut}_{\mathfrak{D}}(\tilde{H})$ out of a torsion part and the non-torsion part. There is an exact sequence

$$\mathbf{Aut}_{\mathbb{Z}}(\text{torsion subgroup of } H_{\mathbb{Z}}) \longrightarrow \mathbf{Aut}(\tilde{H}) \longrightarrow \underbrace{\mathbf{Aut}_{\mathbb{Z}}(H_{\mathbb{Z}}/\text{torsion}) \cap \mathbf{Aut}_{\mathbb{R}}\left(H_{\mathbb{R}}, (\ ,\ )\right)}_{=:G \subset \mathrm{Gl}(\mathbb{R}^k)}.$$

G is the intersection of a discrete group and a compact group (the latter being some kind of orthogonal group, basically), hence finite, rendering $\mathbf{Aut}(\tilde{H})$ finite. □

Proposition 2.2. *Suppose, $\mathcal{S} \in \underline{\mathfrak{Top}}$ is connected and it admits a universal covering $\mathcal{S}'$. Let $(H_{\mathbb{Z}}, F)$ be a polarizable family of Hodge structures over $\mathcal{S}$. If $\underline{F}$ is a sub-local system filtration, there is a finite covering $\pi\colon \mathcal{S}'' \xrightarrow{\pi} \mathcal{S}$ such that $\pi^*(H_{\mathbb{Z}}, F)$ is a constant family of pure Hodge structures on $\mathcal{S}''$.*

Proof. $\pi_1(\mathcal{S}, s_0)$ acts on the stalk $(H_{\mathbb{Z}})_{s_0}$. This action preserves the filtration $\underline{F}$ due to assumption, and it naturally preserves the polarization, as it only comes from a pullback of sheaves. Let $N \coloneqq \ker\left(\pi_1(\mathcal{S}, s_0) \to \mathbf{Aut}(\tilde{H}_{s_0})\right)$.
$\pi_1(\mathcal{S})/N$, due to Proposition 2.1, is finite. We have the corresponding covering $\mathcal{S}'' \xrightarrow{\pi} \mathcal{S}$. As $\pi_1(\mathcal{S}'') \cong N$ acts trivially on the stalks, we conclude that $\pi^*(H_{\mathbb{Z}})$ is constant and thus $\pi^*(H_{\mathbb{C}})$. The constancy of $\pi^*(H_{\mathbb{C}})$ follows from the fact that on $\mathcal{S}''$, the operation of $\pi_1(\mathcal{S}, s_0)$ on the stalks is trivial, and hence the monodromy representation is trivial. □

Proposition 2.3. *For $\mathcal{S} \in \underline{\mathfrak{Top}}$ just as in Proposition 2.2, let $(H_{\mathbb{Z}}, W)$ be a family of mixed Hodge structures on $\mathcal{S}$. Let $H_{\mathbb{Z}}$ and every $(\mathrm{gr}_n^W H_{\mathbb{Z}}, F)$ be polarizable. (Due to finiteness, polarizability*

of all the graded families actually suffices) Suppose that, like in Proposition 2.2, $\underline{F}$ is a locally constant filtration. $(\star)$
Then there exists a finite covering $\mathcal{S}'' \xrightarrow{\pi} \mathcal{S}$ such that $\pi^*(H_\mathbb{Z}, W, F)$ is a constant family of Hodge structures on $\mathcal{S}''$.

Proof. Proposition 2.2 yields a finite covering $\mathcal{S}'' \xrightarrow{\pi} \mathcal{S}$ such that the inverse image of any $\mathrm{gr}_n^W H_\mathbb{Z}$ is locally constant. The action of $\pi_1(\mathcal{S})$ must be compatible with this constancy, so multiplication with the group ring element $1-\gamma \in \mathbb{Z}[\pi_1(\mathcal{S})]$ must be zero on $\mathrm{gr}_n^W = W_n H_\mathbb{Z}/W_{n-1}H_\mathbb{Z} \forall n$, which is equivalent to

$$(1-\gamma)[W_n(H_\mathbb{Z})_{s_0}] \subset W_{n-1}(H_\mathbb{Z})_{s_0}. \tag{2.1}$$

Because of $(\star)$, $1-\gamma$ is an endomorphism of families of Hodge structures and Fact 1.13, we have

$$(1-\gamma)[W_n] \overset{(2.1)}{\subset} W_{n-1}H_\mathbb{Z} \subset \mathrm{Im}(1-\gamma) \overset{\text{strict compatibility}}{=} (1-\gamma)[W_{n-1}]. \tag{$\star\star$}$$

Let m be minimal such that $W_m \neq 0$. From $(\star\star)$, it follows that the $1-\gamma$ homomorphism is zero on W_m and inductively so on all arbitrary W_n, as it is zero on gr^n. Thus $1-\gamma$ is zero on $H_\mathbb{Z}$ which is isomorphic to W_n for $n \gg 0$. Due to the trivial action of $\pi_1(\mathcal{S}'')$, $\pi^*(H_\mathbb{Z})$ is constant on $\mathcal{S}''$. □

3 Hodge structures and cohomology

For now, we have defined the abstract notion of a Hodge structure without actually explaining where it arises from.
Let $X = Y^{\mathrm{an}}$ be the complexification of a smooth complex projective variety; or more generally, X^{an} has to be compact Kähler.

Recall 3.1. A Kähler manifold of dimension n is a holomorphic manifold together with a Hermitian (i.e. symmetric sesquilinear) inner product which comes from a smooth 2-form ω such that ω is closed, i.e. $\mathrm{d}\omega = 0$.

We now fix ω.

Recall 3.2. We have the isomorphism $\mathrm{H}^q(X^{\mathrm{an}}, \mathbb{Q}) \hookrightarrow \mathrm{H}^q(X^{\mathrm{an}}, \mathbb{C}) \cong \mathrm{H}^q_{\mathrm{dR}}(X)$. One defines the Laplacian operator $\Delta\colon \Omega^k(X) \to \Omega^{k-1}(X)$ by $\delta \circ \mathrm{d} + \mathrm{d} \circ \delta$, where δ is $(-1)^{n(k-1)+n+1} \star \mathrm{d}\star$. (Roughly, to a dual-multivector, the Hodge star $\star$ assigns the orthogonal complement.)

Fact 3.3. *In our setup, there is an isomorphism* $\underbrace{\ker \Delta}_{\subset \Omega^k(X)} \cong \mathrm{H}^k_{\mathrm{dR}}(X)$.

Since X is Kähler, a harmonic k-form is a $\mathbb{C}$-linear combination of dual-multivectors $\mathrm{d}z_1 \wedge \cdots z_p \wedge \overline{z}_{p+1}$. Hodge's theorem then states that $\mathrm{H}^k_{\mathrm{dR}}(X)$ decomposes to the sub-vector spaces $V^{p,q}$ of (p,q)-forms and those form a pure Hodge structure of weight k with $\mathrm{H}^k(X) = H_\mathbb{Z}$.

Remark 3.4. Considering

$$H_{\mathbb{Z}} \coloneqq \mathrm{H}^*(X) = \bigoplus_k \mathrm{H}^k(X), H_{\mathbb{C}} = \bigoplus_k \mathrm{H}^k_{\mathrm{dR}}(X), W_k \coloneqq \bigoplus_{j \le k} \mathrm{H}^j(X),$$

one gets a mixed Hodge structure.

Definition & Fact 3.5.

The Hodge-Riemann bilinear form on $\mathrm{H}^k_{\mathrm{dR}}(X)$ is defined by

$$(\alpha, \beta) \mapsto (-1)^{\binom{k}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-k}.$$

A covector/a class $\alpha \in \mathrm{H}^k_{\mathrm{dR}}(X)$ is called primitive $:\Leftrightarrow \omega^{\wedge n-k+1}\alpha = 0$. This leads to the primitive cohomology

$$\mathrm{H}^k_{\mathrm{prim}}(X) \coloneqq \ker\left(\alpha \mapsto \omega^{\wedge(n-k+1)}\alpha\right)$$

.

Decomposing

$$\mathrm{H}^k_{\mathrm{dR}}(X) = \bigoplus_{i \ge \max(k-n,0)} \omega^{k-i} \wedge \mathrm{H}^i_{\mathrm{prim}},$$

the Hodge-Riemann form is a polarization of $\mathbb{R}$-Hodge structures on every summand (and hence on $\mathrm{H}^k_{\mathrm{dR}}(X) \cong \mathrm{H}^k(X;\mathbb{R})$.)

Example 3.6. i) The Tate Hodge structure $\mathbb{R}[-n]$ is a fitting Hodge structure to the cohomology $\mathrm{H}^{2n}(X)$ of a compact complex manifold X.

ii) The generator of $\mathrm{H}^1(\mathbb{C}^\times)$ is the unit circle, which, when integrated, yields the canonical generator of $\mathrm{H}_{\mathrm{dR}}(\mathbb{C}^\times)$ multiplied by $2\pi\mathrm{i}$.

References

[Del71] Pierre Deligne. Théorie de Hodge : II. fr. In: Publications Mathématiques de l'IHÉS 40 (1971), pp. 5–57. URL: `https://www.numdam.org/item/PMIHES_1971__40__5_0/`.

[Kat72] Nicholas M. Katz. Algebraic solutions of differential equations (p-curvature and the Hodge filtration). In: Inventiones mathematicae 18.1 (Mar. 1972), pp. 1–118. URL: `https://doi.org/10.1007/BF01389714`.

[PS08] Chris A.M. Peters and Joseph H.M. Steenbrink. Mixed Hodge Structures. Springer, 2008.