The set $|Y|$ and the untilts of $\mathcal{O}_F$

The following are notes for a talk I gave on 12.11.2025 in the seminar "The Fargues–Fontaine curve and p-adic Hodge theory", see the program. The main reference for this talk was [Ans, §5], and I took some extra inspiration from [Lur, Lectures 1-3, 17]. I added missing details to some of the arguments, but none of this is original work. I hope you enjoy reading this. Comments are always welcome!

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Throughout the talk, p is some prime and E is a finite extension of $\mathbb{Q}_p$. Denote by $\mathcal{O}_E$ the ring of integers of E and let $\pi \in \mathcal{O}_E$ be a uniformiser. Let $q = \#(\mathcal{O}_E/(\pi))$. We will write $\mathbb{F}_q = \mathcal{O}_E/(\pi)$.

That which came before

We recall the important content of the previous talks. To any π-complete $\mathcal{O}_E$-algebra A we can associate its *tilt*

$$A^\flat = \varprojlim_{x \mapsto x^q} A/\pi,$$

i.e. the perfection of its reduction mod π. It comes with a multiplicative map

$$\sharp\colon A^\flat \to A,$$

given by lifting any $(\dots, x_1, x_0) \in A^\flat$ to a sequence $(\tilde{x}_0, \tilde{x}_1, \dots)$ in A and taking the limit $\lim_{i\to\infty} \tilde{x}_i^{q^i}$. Conversely, to any perfect $\mathbb{F}_q$-algebra R we can associate its *(ramified) Witt vectors* $W_{\mathcal{O}_E}(R)$. There is an adjunction between taking Witt vectors and taking flats, whose counit is given by *Fontaine's map*

$$\begin{aligned} \theta_A\colon W_{\mathcal{O}_E}(A^\flat) &\longrightarrow A, \\ \sum_{i=0}^\infty [a_i]\pi^i &\longmapsto \sum_{i=0}^\infty a_i^\sharp \pi^i. \end{aligned}$$

Let C now be some complete, algebraically closed, non-archimedean field extension of E with valuation v_C. Let $\mathcal{O}_C = \{c \in C \mid v(c) \geq 0\}$ be its valuation ring. Then any element of $\mathcal{O}_C$ can be expressed in the form $\sum_{i=0}^\infty c_i^\sharp \pi^i$ with $c_i \in \mathcal{O}_C^\flat$, i.e. $\theta_{\mathcal{O}_C}$ is surjective. Additionally, $\ker\theta_{\mathcal{O}_C}$ is principal and generated by some (non-unit) *distinguished element* $d \in W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)$. This exhibits $\mathcal{O}_C$ as a particular example of a *perfectoid $\mathcal{O}_E$-algebra*, which is an $\mathcal{O}_E$-algebra isomorphic to a quotient $W_{\mathcal{O}_E}(R)/I$, where R is a perfect $\mathbb{F}_q$-algebra and $(W_{\mathcal{O}_E}(R), I)$ is a perfect prism. That means, I is an ideal generated by a distinguished element and $W_{\mathcal{O}_E}(R)$ is I-adically complete. We have seen that these distinguished elements are always of the form $u\pi - [r_0]$, where $u \in W_{\mathcal{O}_E}(R)^\times$ and R is r_0-adically complete.

Finally, $\mathcal{O}_C^\flat$ is a valuation ring with associated valuation $v_C \circ \sharp$. It is complete with respect to this valuation and $\mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed. The central question of this talk is the following:

If we instead fix some complete, algebraically closed, non-archimedean field F with ring of integers $\mathcal{O}_F$, what can we say about the perfectoid $\mathcal{O}_E$-algebras that tilt to $\mathcal{O}_F$?

The answer is in fact that any such perfectoid $\mathcal{O}_E$-algebra is again the ring of integers in a complete, algebraically closed, non-archimedean field, either F or an extension of E. On top of that, the valuation on these perfectoid $\mathcal{O}_E$-algebras is related to the one on $\mathcal{O}_F$ by composition with the sharp map.

Tilting and untilting perfectoid spaces

Let R be a perfect $\mathbb{F}_q$-algebra.

Definition 1. An ***untilt*** of R is a pair (A, i), where A is a perfectoid $\mathcal{O}_E$-algebra and $i\colon A^\flat \cong R$ is an isomorphism. A morphism of untilts $f\colon (A,i) \to (B,j)$ is a morphism $f\colon A \to B$ of $\mathcal{O}_E$-algebras with $f^\flat = j^{-1} \circ i$. Denote the category of untilts of R by $\mathbf{Untilts}(R)$

To determine the untilts of R, we first determine the tilts of the perfectoid algebra $W(R)/I$.

Lemma 2. *Let $(W_{\mathcal{O}_E}(R), I)$ be a perfect prism. Then $(W_{\mathcal{O}_E}(R)/I)^\flat \cong R$. Under this isomorphism, Fontaine's map θ translates to the projection and the sharp map $\sharp$ translates to the reduction of the Teichmüller lift mod I.*

Proof. By a theorem of one of the previous talks, we establish an isomorphism

$$\begin{aligned} R &\longrightarrow (R/IR)^\flat, \\ r &\longmapsto (\ldots, r^{1/q} \bmod IR, r \bmod IR). \end{aligned}$$

Now, we find the isomorphism

$$\varphi\colon R \cong (R/IR)^\flat \cong (W_{\mathcal{O}_E}(R)/((\pi)+I))^\flat \cong (W_{\mathcal{O}_E}(R)/I)^\flat,$$

given explicitly by $\varphi(r) = (\ldots, \overline{[r^{1/q}]}, \overline{[r]})$, where the reductions are taken mod $(\pi) + I$. The second part of the lemma says that the diagram

$$\begin{tikzcd} R \arrow[r, "{[\cdot]}"] \arrow[d, "\varphi"'] & W_{\mathcal{O}_E}(R) \arrow[d, "W_{\mathcal{O}_E}(\varphi)"'] \arrow[dr] & \\ (W_{\mathcal{O}_E}(R)/I)^\flat \arrow[r, "{[\cdot]}"] \arrow[rr, bend right, "\sharp"'] & W_{\mathcal{O}_E}((W_{\mathcal{O}_E}(R)/I)^\flat) \arrow[r, "\theta"] & W_{\mathcal{O}_E}(R)/I \end{tikzcd}$$

commutes. Since the left square obviously commutes, we check that the right triangle commutes. It suffices to check this on Teichmüller lifts $[r] \in W_{\mathcal{O}_E}(R)$. For these elements, commutativity of the diagram means precisely that

$$(\ldots, \overline{[r^{1/q}]}, \overline{[r]})^\sharp = [r] \bmod I,$$

which is immediate from the construction of the sharp map. ■

Remark. Lemma 2 tells us two things:

(1) A perfectoid $\mathcal{O}_E$-algebra A is an untilt of R if and only if it is of the form $W_{\mathcal{O}_E}(R)/I$ for some perfect prism $(W_{\mathcal{O}_E}(R), I)$.

(2) If A is perfectoid, then $(\ker\theta_A, W_{\mathcal{O}_E}(A^\flat))$ is a perfect prism and isomorphic to any other perfect prism defining A. ⌟

Let now (A, i) be an untilt of R. Consider Fontaine's map $\theta_A\colon W_{\mathcal{O}_E}(A^\flat) \to A$ and precompose it with the isomorphism $W_{\mathcal{O}_E}(i^{-1})\colon W_{\mathcal{O}_E}(A^\flat) \to W_{\mathcal{O}_E}(R)$. As $(\ker\theta_A, W_{\mathcal{O}_E}(A^\flat))$ is a perfect prism, so is $(W_{\mathcal{O}_E}(R), \ker(\theta_A \circ W_{\mathcal{O}_E}(i^{-1})))$. We obtain the following identification.

Proposition 3. *The above map induces a bijection*

$$\begin{aligned} \mathbf{Untilts}(R)/\cong &\longrightarrow \{I \subseteq W_{\mathcal{O}_E}(R) \mid (W_{\mathcal{O}_E}(R), I) \text{ perfect prism}\}, \\ (A, i) &\longmapsto \ker(\theta_A \circ W(i^{-1})). \end{aligned}$$

Proof. The above remark already shows that this map is surjective. We will show that it is well-defined and injective. For well-definedness, let $f\colon (A, i) \to (B, j)$ be an isomorphism of untilts. Then $W_{\mathcal{O}_E}(f^\flat) = W_{\mathcal{O}_E}(j^{-1}) \circ W_{\mathcal{O}_E}(i)$ is an isomorphism of $\mathcal{O}_E$-algebras which maps $\ker\theta_A$ onto $\ker\theta_B$. On the contrary, if (A, i) and (B, j) map to the same ideal $I \subseteq W_{\mathcal{O}_E}(R)$. Then the isomorphism $W_{\mathcal{O}_E}(j^{-1} \circ i)$ maps $\ker\theta_A$ onto $\ker\theta_B$. By two-out-of-three, the induced map $f\colon A \to B$ on the cokernels is an isomorphism as well. As θ_B is the counit of the Witt vector-flat adjunction, it follows that $W(f^\flat) = W(j^{-1} \circ i)$ hence by fully faithfulness of $W_{\mathcal{O}_E}$ that $f^\flat = j^{-1} \circ i$. ∎

Remark. Note that, up to isomorphism, there is a unique positive characteristic untilt of R, namely (R, id_R). Under the identification in Proposition 3, it corresponds to the ideal (π).

Untilting certain valuation rings

Let now $F/\mathbb{F}_q$ be some complete, algebraically closed, non-archimedean field with valuation v_F. Let $\mathcal{O}_F = \{x \in F \mid v_F(x) \geq 0\}$ be its associated valuation ring and $\mathfrak{m}_F = \{x \in \mathcal{O}_F \mid v_F(x) > 0\}$ its maximal ideal. Recall that we defined $\mathbb{A}_{\mathrm{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$, as well as $|Y|_{[0,\infty)} = \{I \subseteq \mathbb{A}_{\mathrm{inf}} \mid (\mathbb{A}_{\mathrm{inf}}, I) \text{ perfect prism}\}$ and $|Y| = |Y|_{[0,\infty)} \setminus \{(\pi)\}$. Here, the idea is to interpret the elements of $\mathbb{A}_{\mathrm{inf}}$ to be regular functions on $|Y|_{[0,\infty)}$. By the results of the previous sections, the elements I of $|Y|_{[0,\infty)}$ are in one to one correspondence with the (isomorphism classes of) untilts $\mathbb{A}_{\mathrm{inf}}/I$. Under this correspondence, the elements I of $|Y|$ are precisely the *characteristic zero* untilts. The main result of this talk is now the following:

Theorem 4. *Let $I \in |Y|$. Then $\mathbb{A}_{\mathrm{inf}}/I$ is the valuation ring of a complete, algebraically closed, non-archimedean field extension of E.*

Note in particular that $\mathbb{A}_{\mathrm{inf}}/I$ is a domain, so every $I \in |Y|_{[0,\infty)}$ is a prime ideal. Theorem 4 says practically that every residue field carries a valuation. Before we prove Theorem 4, we prove some auxiliary results.

Recall that any ideal $I \in |Y|$ is generated by an element $d = u\pi - [d_0]$, with $u \in \mathbb{A}_{\mathrm{inf}}^\times$ and $d_0 \in \mathfrak{m}_F \setminus \{0\}$ (as $\mathcal{O}_F$ is d_0-complete if and only if $d_0 \in \mathfrak{m}_F$). Write $D = \mathbb{A}_{\mathrm{inf}}/(d)$.

Lemma 5. *(i) The $\mathcal{O}_E$-algebra D is π-complete.*

(ii) The $\mathcal{O}_E$-algebra D is π-torsion free.

(iii) The map

$$\begin{aligned} D &\longrightarrow D, \\ x &\longmapsto x^p \end{aligned}$$

is surjective.

Proof. (i) As $\mathbb{A}_{\mathrm{inf}}$ is π-complete, the quotient D is *derived* π-complete. As d is distinguished, D has bounded π-torsion and is therefore also π-complete.

(ii) We have the following general fact for commutative rings R: if (r, s) is a regular sequence in R and R is r-complete, then (s, r) is a regular sequence in R.

Note that, as $d_0 \neq 0$, the element $\overline{d} \in \mathbb{A}_{\mathrm{inf}}/(\pi) = \mathcal{O}_F$ is not a zero divisor. Hence, (π, d) is regular in $\mathbb{A}_{\mathrm{inf}}$. By the above general fact, it follows that (d, π) is regular as well. It follows that π is not a zero divisor in $\mathbb{A}_{\mathrm{inf}}/(d)$, in other words that $\mathbb{A}_{\mathrm{inf}}/(d)$ is π-torsion free.

Finally, for completeness, we prove the general fact. We use that an element x is a zero divisor if and only if the multiplication-by-x map is not an injection. Hence, we know that the map $R/(r) \xrightarrow{\cdot\bar{s}} R/(r)$ is an injection. From this, it follows that $R/(r^2) \xrightarrow{\cdot\bar{s}} R/(r^2)$ is an injection as well. After all, suppose $a \in R$ satisfies $as \in (r^2)$. Then $a \in (r)$, say $a = br$. Since r is not a zero divisor in R, it now follows from $brs \in (r^2)$ that $bs \in (r)$, i.e. $b \in (r)$ and $a \in (r^2)$. Repeating that argument, every map $R/(r^n) \xrightarrow{\cdot\bar{s}} R/(r^n)$ is an injection and, passing to the limit, using that R is r-complete, the map $R \xrightarrow{\cdot s} R$ is injective. Hence s is not a zero divisor in R. Now, applying the Nine Lemma to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "\cdot r"] \arrow[d, "\cdot s"] & R \arrow[r] \arrow[d, "\cdot s"] & R/(r) \arrow[r] \arrow[d, "\cdot \bar{s}"] & 0 \\
0 \arrow[r] & R \arrow[r, "\cdot r"] \arrow[d] & R \arrow[r] \arrow[d] & R/(r) \arrow[r] \arrow[d] & 0 \\
 & R/(s) \arrow[r, "\cdot \bar{r}"] \arrow[d] & R/(s) \arrow[r] \arrow[d] & R/(r,s) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

yields that the map $R/(s) \xrightarrow{\cdot\bar{r}} R/(s)$ is injective, i.e. $\bar{r} \in R/(s)$ is not a zero divisor.

(iii) (Sketch) It suffices to prove the statement in case $E = \mathbb{Q}_p$ and $\pi = p$. The general case can be derived from this using the norm map. We also assume $p \neq 2$. The case $p = 2$ is resolved by changing all twos in the proof by threes.

We use the description of the sharp map from Lemma 2. It is clear that every element $z^\sharp$ admits the p-th root $(z^{1/p})^\sharp$, given that F is algebraically closed and $\mathcal{O}_F$ is its valuation ring. Therefore, it suffices to prove the statement up to multiplication with elements of the form $z^\sharp$ with $z \in \mathcal{O}_F$. Given that $d_0^\sharp = up \bmod d$ and D is p-complete, it is also $d_0^\sharp$-complete. This means that we can write any $x \in D$ in the form $x = \sum_{i=0}^{\infty} x_i^\sharp \cdot (d_0^\sharp)^i$, where $v_F(x_i) < v_F(d_0)$. We may assume up to multiplication with sharps that $x_0^\sharp \neq 0$. Then, multiplying with $(x_0^\sharp)^{-1}$ (note, crucially, that this product is still an element of D), we may assume that $x \in 1 + (p, \mathfrak{m}_F^\sharp)$. Now, by explicit computation, we can find some $z \in \mathcal{O}_F^\times$ such that $x \equiv z^\sharp \bmod p^2$. This computation uses the fact that F is algebraically closed. Multiplying with $(z^{-1})^\sharp$, we may assume that $x \in 1 + (p^2)$, all whose elements have a p-th root. ■

Finally, we can prove Theorem 4.

Proof of Theorem 4. Consider the multiplicative map $\sharp\colon \mathcal{O}_F \to D$. Recall that it sends any $x \in \mathcal{O}_F$ to $x^\sharp \bmod d$. Note that his map only sends zero to zero. By Lemma 5.(iii), $\sharp$ is surjective. After all, if $y \in D$, then all y^{1/q^i} are in D, hence $(\ldots, y^{1/q} \bmod \pi, y \bmod \pi)^\sharp = y$. Given that $d_0^\sharp = u\pi \bmod d$, the map extends to a surjective, multiplicative map $\sharp\colon \mathcal{O}_F[\frac{1}{d_0}] \to D[\frac{1}{\pi}]$. Note that $D[\frac{1}{\pi}]$ is *not* the zero ring, as D is π-torsion free by 5.(ii).

We now deduce a lot of structure on D and $D[\frac{1}{\pi}]$ from that on $\mathcal{O}_F[\frac{1}{d_0}]$. For instance, $D[\frac{1}{\pi}]$ is a field as $\mathcal{O}_F[\frac{1}{d_0}] = F$ is: any nonzero $y = x^\sharp \in D$ has inverse $(x^{-1})^\sharp$. This means in particular that D is a domain and $D[\frac{1}{\pi}] = \mathrm{Frac}(D)$. Also, for any $y = x^\sharp \in D[\frac{1}{\pi}]$, either x or x^{-1} is in $\mathcal{O}_F$, hence either y or $y^{-1} = (x^{-1})^\sharp$ is in D. Therefore, D is a valuation ring.

We will go one step further and explicitly determine the valuation on D. To do this, we show that $x_1^\sharp \mid x_2^\sharp$ in D *if and only if* $x_1 \mid x_2$ in $\mathcal{O}_F$. Since the 'if' part is clear, we prove the 'only if' part, so let $x_1, x_2 \in \mathcal{O}_F \setminus \{0\}$ be such that $x_1^\sharp \mid x_2^\sharp$ and assume $x_1 \nmid x_2$ (the case were either element is zero can immediately be observed to be true). Then, as $\mathcal{O}_F$ is a valuation ring, $x_1 \mid x_2$, say $x_2 = tx_1$, where $t \in \mathfrak{m}_F$. This means that $x_2^\sharp = t^\sharp x_1^\sharp$. It follows by the assumptions that $t^\sharp$ is a unit. However, the sharp map induces an isomorphism $\mathcal{O}_F/(d_0) \to D/(\pi)$. Since t is a nilpotent element on the left but $t^\sharp$ is a unit on the right, we obtain a contradiction.

We now define the map

$$
\begin{aligned}
v_D\colon D &\longrightarrow \mathbb{R} \cup \{\infty\}, \\
x^\sharp &\longmapsto v_F(x).
\end{aligned}
$$

Note that this map is well-defined: if any two elements x, x' map to the same element in D under $\sharp$, then they must differ by a unit. Also note that $v_D(y) \leq v_D(y')$ if and only if $y \mid y'$. One can now show that v_D in fact extends to a valuation on $D[\frac{1}{\pi}]$ with valuation ring D.

Finally it remains to show that $D[\frac{1}{\pi}]$ is an algebraically closed and complete field extension of E. As $\pi \neq 0$ in D and $\mathcal{O}_E[\frac{1}{\pi}] = E$, it is immediately clear that $D[\frac{1}{\pi}]$ is an E-algebra (i.e. a field extension). Completeness with respect to the valuation topology induced by v_D follows from the fact that its valuation ring D is π-adically complete (by Lemma 5.(i)) and $\pi \neq 0$ has valuation $v_D(\pi) = v_F(d_0) > 0$. To prove that $D[\frac{1}{\pi}]$ is algebraically closed, we consider an irreducible polynomial

$$P = T^m + b_{m-1}T^{m-1} + \cdots + b_0 \in D[T]$$

and show that it has zero. Given that $D/(\pi) = \mathcal{O}_F/(d_0)$, let $Q \in \mathcal{O}_F[T]$ be a polynomial such that $P \equiv Q$ in $D/(\pi)[T] = \mathcal{O}_F/(d_0)[T]$. Then Q has a zero in $\mathcal{O}_F$ (since the constant term of Q is, up to negatives, the product of all zeroes), call it z_0. It follows that $P(T + z_0^\sharp)$ is irreducible and must have constant term $P(z_0^\sharp)$ divisible by π. Let $c_1 \in D$ be such that $v_D(c_1) = \frac{v_D(P(z_0^\sharp))}{m} \geq \frac{v_D(\pi)}{m}$. Such an element exists as $v_D(D[\frac{1}{\pi}]^\times) = v_F(F^\times)$ and $v_F(F^\times)$ is divisible as F is algebraically closed. Define

$$P_1 = c_1^{-m}P(z_0^\sharp + c_1T).$$

It is still irreducible and it therefore has coefficients in D, as the leading and constant coefficient are in D. Repeating the above process, we can find $z_1 \in \mathcal{O}_F$ with $v_D(P_1(z_1^\sharp)) \geq v_D(\pi)$, which means that $v_D(P(z_0^\sharp + c_1z_1^\sharp)) \geq mv_D(c_1) + v_D(\pi) \geq 2v_D(\pi)$. Note that $v_D(c_1z_1^\sharp) \geq \frac{v_D(\pi)}{m}$. Repeating this one more time yields a $c_2 \in D$ with $v_D(c_2) \geq \frac{v_D(\pi)}{m}$ and a $z_2 \in \mathcal{O}_F$ such that $v_D(P(z_0^\sharp + c_1z_1^\sharp + c_1c_2z_2^\sharp)) \geq 3v_D(\pi)$. Note that $v_D(c_1c_2z_2^\sharp) \geq 2\frac{v_D(\pi)}{m}$. We can see now that repeating this process yields a sequence $(s_n)_{n\geq 0}$ in D with $v_D(s_n - s_{n-1}) \geq n\frac{v_D(\pi)}{m}$ and $v_D(P(s_n)) \geq (n+1)v_D(\pi)$. Hence, this process yields a zero of P in the limit. ■

References

[Ans] J. Anschütz. *Lectures on the Fargues–Fontaine curve*. `https://janschuetz.perso.math.cnrs.fr/skripte/vorlesung_the_curve.pdf`. Lecture notes.

[Lur] J. Lurie. *The Fargues–Fontaine curve*. `https://www.math.ias.edu/~lurie/205.html`. Lecture notes.