The definition of the curve

§ 0. Recap of def^{n}s + notation from earlier talks

- $E/\mathbb{Q}_p$ finite, $\mathcal{O}_E$ = ring of integers, $\pi \in \mathcal{O}_E$ uniformiser
  $\mathcal{O}_E/(\pi) \cong \mathbb{F}_q$ ($q = p^f$ some $f \geq 1$)
- We fix throughout a complete, algebraically closed non-archimedean (NA) field ext^n F of $\mathbb{F}_q$.
  We denote by $v_F : F^\times \longrightarrow \mathbb{R}$ its valuation, $|-|_F = q^{-v_F(-)}$ abs. value, $\mathcal{O}_F = \{x \in F \mid v_F(x) \geq 0\}$ valuation ring and $\mathfrak{m}_F = \{x \in F \mid v_F(x) > 0\}$ max^l ideal
  We also fix $\varpi \in \mathcal{O}_F$ a "pseudo-uniformiser," i.e. s.t $0 < |\varpi|_F < 1$

e.g. $F = \widehat{\overline{\mathbb{F}_q((t))}}$ (i.e. completion of alg. closure) with $\varpi = t$.

- $A_{inf} = W_{\mathcal{O}_E}(\mathcal{O}_F) = \left\{ \sum_{n \geq 0} [x_n] \pi^n \mid x_n \in \mathcal{O}_F \ \forall n \geq 0 \right\}$
- $|Y| = \left\{ J \subsetneq A_{inf} \text{ ideal} \;\middle|\; \begin{array}{l} (A_{inf}, J) \text{ is a perfect prism,} \\ J \neq (\pi) \end{array} \right\}$
  For each $y \in |Y|$, we denote by $\mathfrak{p}_y \subseteq A_{inf}$ the corresponding (prime) ideal and recall that $\mathfrak{p}_y = (\xi_y)$ where $\xi_y = \pi - [a]$ for some $0 \neq a \in \mathfrak{m}_F$.
- We denote by $C_y := (A_{inf}/\mathfrak{p}_y)[\frac{1}{\pi}]$
  $= \mathrm{Frac}(A_{inf}/\mathfrak{p}_y)$

for each $y \in Y$. For $f \in \mathbb{A}_{\inf}$, we then write $f(y)$ for the image of f in C_y under the quotient map $\theta_y : \mathbb{A}_{\inf} \twoheadrightarrow \mathbb{A}_{\inf}/p_y \subseteq C_y$.

Slogan: Elements of $\mathbb{A}_{\inf}$ should be thought of as functions on $|Y|$.

•) For each $y \in |Y|$ there is a unique well-defined valuation v_y on C_y determined by $v_y(\theta_y([x])) = v_F(x)$ for all $x \in \mathcal{O}_F$. Under different notation, this v_y was constructed in Dan's talk to prove that C_y is a NA field ext^n of E

•) Last time: There is a "metric" d on $|Y|$ defined by $d(y_1, y_2) = v_{y_2}(\theta_{y_2}(\xi_{y_1}))$, and we also set $d(y, 0) = v_y(\theta_y(\pi))$. Unpacking the def^{n}s, and writing $\xi_{y_i} = \pi - [a_i]$ $(i=1,2)$, $\xi_y = \pi - [a]$, then $d(y_1, y_2) = v_{y_2}(\theta_{y_2}([a_1] - [a_2]))$ and $d(y, 0) = v_F(a)$

Idea: d was not actually a metric. Instead the map $(y_1, y_2) \longmapsto q^{-d(y_1, y_2)}$ is an actual (ultra)metric (Some authors do not differentiate between norms/absolute values and valuations, and here it is similar: d is a bit like a valuation on $|Y|$)

Similarly, the "distance to 0" should

be $q^{-d(y,0)} = q^{-v_F(a)} = |a|_F \in (0,1)$

Updated slogan: ·) The set $|Y|$ should be thought of as a punctured unit disk $\mathbb{D}^* = \{0 < |x| < 1\}$ over some NA field K

·) A_{inf} should be thought of as analogous to $\mathcal{O}_K[[T]]$, which embeds into the ring of bounded holomorphic functions on $\mathbb{D}^*$, with π playing the role of T (i.e. it should be thought of as a variable!)

§1. Heuristics for the construction

Recall: There is a bijection

$$\mathrm{Untilts}_{\mathrm{char}\,0}(\mathcal{O}_F)/\simeq \xrightarrow{1:1} |Y|$$

$$(A, \iota : A^{\flat} \xrightarrow{\sim} \mathcal{O}_F) \longmapsto \mathrm{Ker}\big(\underbrace{W_{\mathcal{O}_E}(\mathcal{O}_F)}_{=A_{inf}} \xrightarrow{W_{\mathcal{O}_E}(\iota^{-1})} W_{\mathcal{O}_E}(A^{\flat}) \xrightarrow{\theta} A\big)$$

However: The LHS has a bit of redundancy because if $(A, \iota) \in \mathrm{Untilts}(\mathcal{O}_F)$ then so is $(A, \varphi^j \circ \iota)$ for any $j \in \mathbb{Z}$ (and $\varphi \equiv$ Frobenius on $\mathcal{O}_F$)

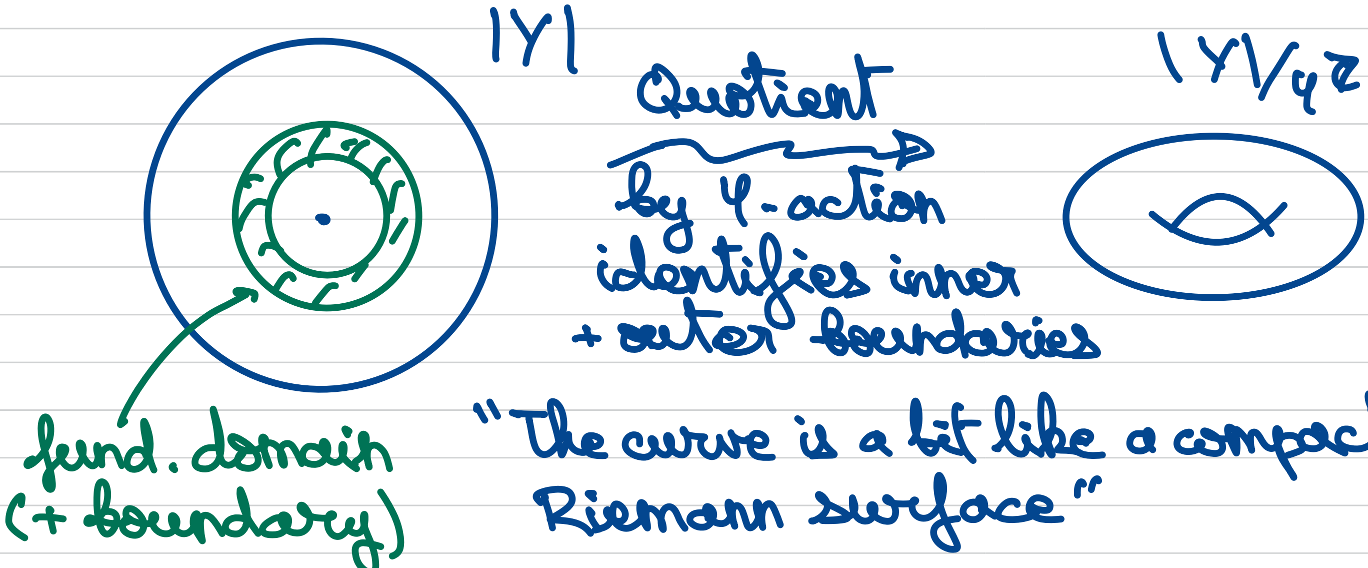
This translates under above action to the usual Frobenius action on (ideals of) A_{inf}, i.e. sending $\sum_n [x_n]\pi^n$ to $\sum_n [x_n^q]\pi^n$.

Write $|Y|/\varphi^{\mathbb{Z}}$ for the quotient by above Frobenius action.

Goal: Construct a scheme X whose closed points are in natural bijection with $|Y|/\varphi^{\mathbb{Z}}$

Heuristic picture: Note that for $y \in |Y|$ we have $\varphi(\xi_y) = \pi - [a^q]$ and so $d(\varphi(y), 0) = v_F(a^q)$
$= q \cdot v_F(a)$

So in our distance $q^{-d(y,0)}$, we see that φ "shifts" elements of $\mathbb{D}^*$ by raising the radius to the power q. A fundamental domain should therefore look like an annulus



"The curve is a bit like a compact Riemann surface"

We also would like our object to look like a $\mathbb{P}^1$ in a suitable sense. For this, we will:

- Construct an analogue B of "holomorphic functions on $\mathbb{D}^*$" with Frobenius action.
- Take $\mathrm{Proj}\left(\bigoplus_{d \in \mathbb{Z}} B^{\varphi = \pi^d}\right)$ as our scheme, where $B^{\varphi = \pi^d} = \{ f \in B \mid \varphi(f) = \pi^d f \}$.

§2. The rings B_I

Def^n 1: We let
$$B^b = A_{\inf}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$$
$$= \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \,\middle|\, x_n \in F \ \forall n, \sup_n |x_n|_F < \infty \right\}$$
Here, $[x_n] := [\varpi]^{-d}[\varpi^d x_n]$ for $d \geq 0$ s.t. $\varpi^d x_n \in \mathcal{O}_F$

Analogy: To get all bounded holomorphic functions on D^* from $\mathcal{O}_K[\![T]\!]$, need to invert T (as $0 \notin D^*$) and ϖ_K (to allow bounds bigger than 1).

Recall from last time: •) For each $r \geq 0$ there is a valuation
$$v_r : A_{\inf} \longrightarrow \mathbb{R}$$
$$f = \sum_{n \geq 0} [x_n]\pi^n \longmapsto \inf_n (v_F(x_n) + rn)$$
•) For $f \in A_{\inf}$, can define $\mathrm{Newt}(f)$ via
$$\mathcal{L}(\mathrm{Newt}(f))(r) = \begin{cases} v_r(f) & \text{if } r \geq 0 \\ -\infty & \text{if } r < 0 \end{cases}$$
These def^{n}s extend straightforwardly to B^b

Def^n 2: (a) For $r > 0$, we define the r-Gauss norm on B^b as $|f|_r := q^{-v_r(f)} = \sup_n q^{-rn}|x_n|_F$

(b) For $I \subseteq (0,\infty)$ an interval (not nec. open nor closed), we let B_I = completion of B^b w.r.t. all $|-|_r$ with $r \in I$.

Rmk 3: •) Since v_r is a valuation, we have $|fg|_r = |f|_r \cdot |g|_r$ for all $f, g \in B^b$ and $r > 0$.
Hence B_I is a ring.
•) We should think of B_I as holom. functions on $|Y_I| = \{y \in Y \mid d(y,0) \in I\}$. When $I = [a,b]$, this is analogous to an annulus $\{q^{-b} \leq |x| \leq q^{-a}\} \subseteq \mathbb{D}^*$.
•) If $I_1 \subseteq I_2$ then $id: B^b \to B^b$ extends by universal property of completion to $B_{I_2} \to B_{I_1}$ ("restricting to a smaller annulus")
•) For $f \in B^b$, have that $r \mapsto v_r(f)$ is concave
$\Rightarrow$ if $a \leq r \leq b$ then $|f|_r \leq \sup\{|f|_a, |f|_b\}$
(If $r = ta + (1-t)b$ then concave property implies LHS $\leq |f|_a^t \cdot |f|_b^{1-t}$)
Hence, if $I = [a,b]$ then B_I is the completion of B^b w.r.t. $\|-\|_I := \sup(|-|_a, |-|_b)$.
In particular, B_I is a Banach algebra.
•) For gen^l I, have $B_I = \varprojlim_{[a,b] \subseteq I} B_{[a,b]}$ (this is in gen^l not Banach but instead a Fréchet algebra)

A more explicit description of B_I in enough cases

Note: If $I = [a,b]$ then $\|fg\|_I \leq \|f\|_I \cdot \|g\|_I$ (but no longer $=$ in gen^l). By gen^l theory, it follows

that $B_I \cong \hat{R}^\circ_I[\frac{1}{\pi}]$ where $R^\circ_I = \{f \in B^b \mid \|f\|_I \le 1\}$ ← π-adic completion
(cf. Appendix at end of these notes)
In gen^l, unclear how to describe R°_I. But if we assume that $a, b \in v_F(F^\times)$ then we can.

Lemma 4: Let $I = [a,b] \subseteq (0,\infty)$ and suppose $\exists\, x_a, x_b \in \mathcal{O}_F$ s.t. $v_F(x_a) = a$ and $v_F(x_b) = b$. Then $R^\circ_I = A_{\inf}\left[\frac{[x_b]}{\pi}, \frac{\pi}{[x_a]}\right]$ and so

$$B_I \cong \widehat{A_{\inf}\left[\frac{[x_b]}{\pi}, \frac{\pi}{[x_a]}\right]}\left[\frac{1}{\pi}\right]$$

Proof: By def^n, $|f|_r \le 1$ $\forall f \in A_{\inf}$, $\forall r > 0$. Moreover, one computes

$$\left|\frac{[x_b]}{\pi}\right|_b = \left|\frac{\pi}{[x_a]}\right|_a = 1$$

and

$$\left|\frac{[x_b]}{\pi}\right|_a = \left|\frac{\pi}{[x_a]}\right|_b = q^{a-b} < 1$$

$$\Rightarrow A_{\inf}\left[\frac{[x_b]}{\pi}, \frac{\pi}{[x_a]}\right] \subseteq R^\circ_I.$$

For reverse inclusion, pick $f = \sum_{n \gg -\infty} [x_n]\pi^n \in R^\circ_I$. By def^n of B^b, $\exists\, m \gg 0$ s.t. $x_a^m \cdot x_n \in \mathcal{O}_F$ for all n, and

$$f = \sum_{n<m} [x_n]\pi^n + \underbrace{\left(\sum_{n \ge 0} [x_{n+m} \cdot x_a^m]\pi^n\right)\left(\frac{\pi}{[x_a]}\right)^m}_{\in A_{\inf}\left[\frac{[x_b]}{\pi}, \pi/[x_a]\right] \checkmark}$$

$\Rightarrow$ WLOG the Teichmüller expression for f is finite.

Then $|f|_a, |f|_b \leq 1$ imply that for each n we have $|x_n|_F \cdot q^{-na}, |x_n|_F \cdot q^{-nb} \leq 1$.

Thus $x_n \cdot x_a^n, x_n \cdot x_b^n \in \mathcal{O}_F \ \forall n$.

For $n \leq 0$, this implies

$$[x_n]\pi^n = [x_n \cdot x_b^n] \cdot \left(\frac{[x_b]}{\pi}\right)^{-n}$$

For $n \geq 0$, this implies

$$[x_n]\pi^n = [x_n \cdot x_a^n] \cdot \left(\frac{\pi}{[x_a]}\right)^n$$

Together these give $f \in A_{\inf}\left[\frac{[x_b]}{\pi}, \frac{\pi}{[x_a]}\right]$ as required. $\square$

Note: $v_F(F^\times) \subseteq \mathbb{R}$ is dense so the above covers "most" cases of closed intervals.

§ 3. Newton polygons for B_I

Back to general $I = [a,b]$: all v_r with $r \in I$ extend to B_I by continuity (e.g. $v_r(f) = -\log_q |f|_r$)

<u>Recall:</u> Let $f \in B^b$. Then the map

$$r \longmapsto \mathcal{L}(\mathrm{Newt}(f))(r)$$

is piecewise linear, concave, with integer slopes (this held for $A_{\inf} \Rightarrow$ also for B^b)

<u>Want:</u> Extend this to B_I by taking limits

but it is not clear a priori that these properties will still hold. Issue is also that only have v_r for $r \in I$ now.

Note: To see slopes are integers, note that $\sup_n |x_n|_F < \infty \implies v_r(f) = \inf_n (v_F(x_n) + rn)$ is achieved for finitely many n's, say $n_0 < \cdots < n_k$. Then for $\varepsilon > 0$ small enough one has

$$v_{r+\varepsilon}(f) = v_r(f) + \varepsilon n_0$$
$$v_{r-\varepsilon}(f) = v_r(f) - \varepsilon n_k$$

$\ulcorner$ essentially, this holds because $\exists N \gg 0$ s.t. $v_r(f) = \min_{n \leq N} (v_F(x_n) + rn)$

and $v_{r \pm \varepsilon}(f) = \min_{n \leq N} (v_F(x_n) + rn \pm \varepsilon n)$ $\lrcorner$

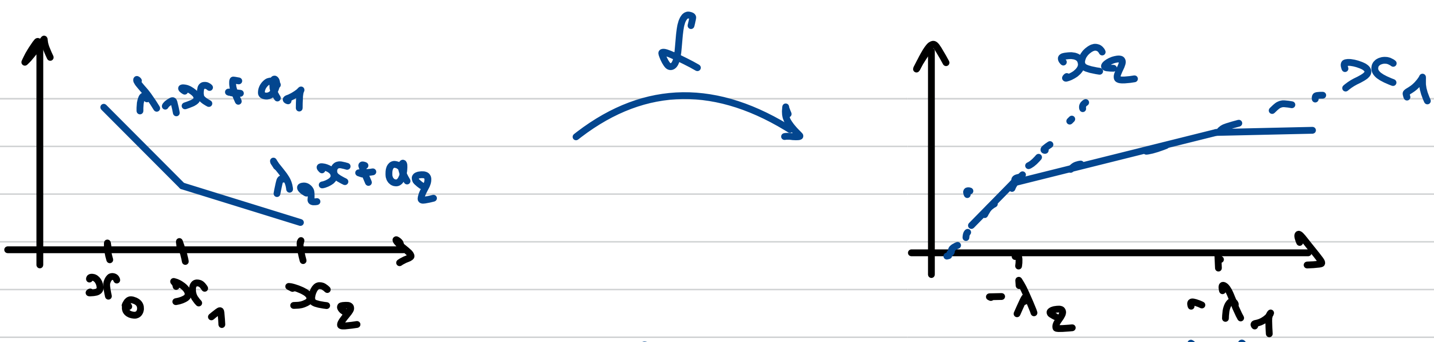
Consequently, can define:

-) $\partial_+ v_r(f) :=$ right derivative of $\mathcal{L}(\mathrm{Newt}(f))$ at r

-) $\partial_- v_r(f) :=$ left derivative of $\mathcal{L}(\mathrm{Newt}(f))$ at r

One has in above notation $\partial_- v_r(f) = n_k$ and $\partial_+ v_r(f) = n_0$, i.e. these are integers and $\partial_- v_r(f) \geq \partial_+ v_r(f)$.

Significance of this notion: If $\lambda < 0$ is a slope of $\mathrm{Newt}(f)$ then $\partial_- v_{-\lambda}(f) - \partial_+ v_{-\lambda}(f)$ is the multiplicity of λ. This is best seen by the following picture taken from Dzoara's talk.



i.e $\partial_- v_{-\lambda_2}(f) - \partial_+ v_{-\lambda_2}(f) = x_2 - x_1$ in picture
$= \text{multiplicity of } \lambda_2$

The key result we need is:

Prop^n 5: Suppose $(f_n)_{n\geq 0}$ is a Cauchy sequence in B^b w.r.t every $|-|_r$ with $r \in I$, such that f_n does not tend to 0 in B_I. Then $\exists N \gg 0$ s.t. for every $n \geq N$ and every $r \in I$,
$$v_r(f_n) = v_r(f_N),\ \partial_+ v_r(f_n) = \partial_+ v_r(f_N)$$
and $\partial_- v_r(f_n) = \partial_- v_r(f_N)$.

Proof: We first need a weaker statement

Claim: Suppose $r \in I$ is s.t. $|f_n|_r$ does not tend to 0. Then $\exists N \gg 0$ s.t $v_r(f_n) = v_r(f_N)$, $\partial_+ v_r(f_n) = \partial_+ v_r(f_N)$ and $\partial_- v_r(f_n) = \partial_- v_r(f_N)$ $\forall n \geq N$.

Proof of claim: As (f_n) is Cauchy and since $|f_n|_r \not\to 0$, the sequence $v_r(f_n)$ converges to some $y \in \mathbb{R}$ (with $y = -\log_q(\lim_{n\to\infty} |f_n|_r)$).
Pick $\varepsilon > 0$ and $\tilde{N} \gg 0$ s.t. $v_r(f_n) \in (y-\varepsilon, y+\varepsilon)$ for all $n \geq \tilde{N}$. As (f_n) is Cauchy, we may

now pick $N \geq \tilde{N}$ s.t. $v_r(f_n - f_N) > y + \varepsilon$ for all $n \geq N$. As v_r is a valuation we deduce that

$v_r(f_n) = v_r(f_N) \quad \forall n \geq N$.

Since $v_r(f_n - f_N) > v_r(f_N)$, continuity of $r \mapsto v_r(f)$ gives that for some $\tilde{\varepsilon} > 0$ we have $v_t(f_n - f_N) > v_t(f_N)$ for all $t \in (r - \tilde{\varepsilon}, r + \tilde{\varepsilon})$

This implies as above that $v_t(f_n) = v_t(f_N)$ for all $n \geq N$. It therefore follows that

$$\partial_+ v_r(f_n) = \partial_+ v_r(f_N)$$
$$\partial_- v_r(f_n) = \partial_- v_r(f_N)$$

for all $n \geq N$ as required.

$\square$

Not sure we need this

We may now prove the result. WLOG $f_n \neq 0 \; \forall n$ and by assumption $\|f_n\|_I \not\to 0$, so that one of $|f_n|_a$, $|f_n|_b$ does not tend to 0. We assume $|f_n|_a \not\to 0$, the other case being completely similar.

Claim $\Rightarrow \exists M \gg 0$ s.t. $v_a(f_n) = v_a(f_M) = s \in \mathbb{R}$

$$\partial_+ v_a(f_n) = \partial_+ v_a(f_M) = k \in \mathbb{Z} \quad \forall n \geq M.$$

As $r \mapsto v_r(f_n)$ is concave, it follows that $v_r(f_n) \leq s + k(r-a)$ for all $r \geq a$

In part., $r \mapsto v_r(f_n)$ is bounded above by $s' := \max\{s, s + k(b-a)\}$ on $[a,b]$.
(f_n) is Cauchy $\Rightarrow$ $\exists N \geq M$ s.t. $\|f_n - f_N\|_I < q^{-s'}$,
i.e. $v_r(f_n - f_N) > s'$ $\forall r \in I$,
for all $n \geq N$.

As $v_r(f_n) \leq s'$ this implies that $v_r(f_n) = v_r(f_N)$ for all $n \geq N$. The statement about derivatives follows exactly as in the proof of the Claim. $\square$

Side comment: From assumptions, only know that $|f_n|_r \not\to 0$ for some $r \in I$, but Prop^n 5 implies that this holds for every $r \in I$.

Corollary 6: Let $I = [a,b]$ and $0 \neq f \in B_I$.
The function $I \to \mathbb{R}$, $r \mapsto v_r(f)$, is well-defined, piecewise linear and concave with integer slopes.

Proof: We may write $f = \lim_n f_n$ for some Cauchy sequence (f_n) in B^b for $\|\cdot\|_I$-topology. The result follows immediately from the corresponding statement in B^b and Prop^n 5. $\square$

Def^n 7: •) Let $I=[a,b]$ and $0\neq f\in B_I$.
We define $\mathrm{Newt}_I^\circ(f)$ to be the convex function with

$$\mathcal{L}(\mathrm{Newt}_I^\circ(f))(r)=\begin{cases} v_r(f) & \text{if } r\in I\\ v_a(f)+(r-a)\partial_- v_a(f) & \text{if } 0\le r\le a\\ v_b(f)+(r-b)\partial_+ v_b(f) & \text{if } r\ge b\\ -\infty & \text{if } r<0\end{cases}$$

and $\mathrm{Newt}_I(f)$ to be the subset of graph of $\mathrm{Newt}_I^\circ(f)$ with slopes in $-I$ (and we set $\mathrm{Newt}_I(f)=\emptyset$ if no such slopes exist).
Finally, we set $\mathrm{Newt}_I(0)=\emptyset$.
•) For gen^l $I\subseteq(0,\infty)$ and $f\in B_I$, we define

$$\mathrm{Newt}_I(f)=\bigcup_{[a,b]\subseteq I}\mathrm{Newt}_{[a,b]}(f)$$

Rmk 8: Suppose $I=[a,b]$ and $f\in B_I$.
•) The point of the above construction is that $\mathrm{Newt}_I(f)=\mathrm{Newt}_I(g)$ for some $g\in B^b$ (cf. Prop^n 5). Hence $\mathrm{Newt}_I(f)$ has finitely many slopes and integral break points.
•) The formula for $\mathcal{L}(\mathrm{Newt}_I^\circ(f))$ is designed to ensure that the multiplicities of the slopes $-a$ and $-b$ are correct.
•) As always, we have that

$$\mathrm{Newt}_I^\circ(f) * \mathrm{Newt}_I^\circ(g) = \mathrm{Newt}_I^\circ(fg)$$
for all $f, g \in B_I$.

§4. The rings B and P

Recall: we are aiming for a space whose points are $|Y|/\varphi^{\mathbb{Z}}$

$\rightsquigarrow$ first need to consider how Frobenius interacts with the Gauß norms.

For $f \in B^b$ one has

$$v_r(\varphi(f)) = \sup_n \left(v_F(x_n^q) + rn \right)$$
$$= q\, v_{r/q}(f)$$
$$\text{i.e. } |\varphi(f)|_r = \left(|f|_{r/q} \right)^q$$

Hence, if $I = [a,b]$ then φ induces a continuous isomorphism $\varphi : B_I \xrightarrow{\sim} B_{qI}$.

Def^n 9: (a) Let $B := B_{(0,\infty)} = \varprojlim_{I = [a,b] \subseteq (0,\infty)} B_I$

Taking limit of $\varphi : B_I \to B_{qI}$ over all I, one gets a continous automorphism $\varphi : B \xrightarrow{\sim} B$.

(b) We let $P = \bigoplus_{d \in \mathbb{Z}} B^{\varphi = \pi^d}$, a $\mathbb{Z}$-graded ring, and put $X = \mathrm{Proj}(P)$ (the <u>schematic FF-curve</u>)

We first note that taking these completions of B^b is really required in order to get an interesting P:

Lemma 10: We have $(B^b)^{\varphi=\pi^d} = \begin{cases} E & \text{if } d=0 \\ 0 & \text{if } d\neq 0 \end{cases}$

Proof: ① Case $d=0$: Let $f = \sum_{n \gg -\infty} [x_n]\pi^n \in B^{\varphi=1}$.

Then $\sum [x_n^q]\pi^n = \varphi(f) = f = \sum [x_n]\pi^n$

$\iff x_n^q = x_n \ \forall n \iff x_n \in \mathbb{F}_q \ \forall n$

$\iff f \in E$ ✓

② Case $d \neq 0$: Assume $f \in (B^b)^{\varphi=\pi^d}$ and let $x \in \mathbb{R}$.

Then $q\,\mathrm{Newt}(f)(x) = \mathrm{Newt}(\varphi(f))(x)$

$= \mathrm{Newt}(\pi^d f)(x)$

$= \mathrm{Newt}(f)(x-d)$

(Recall that $\mathrm{Newt}(f)$ is the convex function lying below $\{(n, v_F(x_n))\}_{n\in\mathbb{Z}}$)

$\Rightarrow q^n \mathrm{Newt}(f) = \mathrm{Newt}(f)(x-nd) \quad \forall n \geq 1$ (*)

•) If $d>0$ then for $n \gg 0$ one has $\mathrm{Newt}(f)(x-nd) = \infty$ and (*) now implies $\mathrm{Newt}(f)(x) = \infty$. As x is arbitrary, we deduce $f=0$

•) If $d<0$, we take $x_0 \ll 0$ s.t.

$\mathrm{Newt}(f)(x_0) \geq \mathrm{Newt}(f)(x)$

As $\mathrm{Newt}(f)$ is decreasing, for $n \gg 0$ we get

$\mathrm{Newt}(f)(x) \geq \mathrm{Newt}(f)(x_0 - nd)$

$= q^n\,\mathrm{Newt}(f)(x_0)$ by (*)

$\geq q^n\,\mathrm{Newt}(f)(x)$

We deduce that $\mathrm{Newt}(f) \equiv \infty$ again and so that $f = 0$ $\square$

We finish by describing some elts of P and B.

Lemma 11: (i) Suppose that $(x_n)_{n \in \mathbb{Z}}$ is a sequence in F s.t. $\lim_{|n| \to \infty} (v_F(x_n) + nr) = \infty$ $\forall r > 0$. Then $f = \sum_{n \in \mathbb{Z}} [x_n] \pi^n$ converges in B.

(ii) There is a well-defined map

$m_F \longrightarrow (B^b)^{\varphi = \pi}$, $a \longmapsto f_a := \sum_{n \in \mathbb{Z}} [a^{q^{-n}}] \pi^n$

Proof: (i) The cond^{n}s ensure that $|[x_n]\pi^n|_r \longrightarrow 0$ as $n \longrightarrow \infty$ for all $r > 0$, which is all we need for the partial sums to be Cauchy w.r.t all $|\cdot|_r$.

(ii) We have $v_F(a^{q^{-n}}) + nr = q^{-n} v_F(a) + nr \to \infty$ as $|n| \longrightarrow \infty$, so that f_a is well-def^d by (i).

Further, $\varphi(f_a) = \sum_n [a^{q^{-n+1}}] \pi^n$

$= \pi \sum_n [a^{q^{-(n-1)}}] \pi^{n-1}$

$= \pi f_a$ as required. $\square$

Rmk 12: (a) It is not known whether:

- all elements of B can be written as in (i);
- elts of B as in (i) have a unique such expression; or
- elts of B as in (i) are stable under $+$ or $\cdot$.

(b) We can construct more generally elts of $B^{\varphi=\pi^d}$ for $d>0$. The idea is to look for an expression $f=\sum_{n\in\mathbb{Z}}[x_n]\pi^n$ as in (i). Since $\varphi(f)=\sum_n [x_n^q]\pi^n$, for this to be in $B^{\varphi=\pi^d}$ it suffices (but maybe is not necessary) to have $x_{n-d}=x_n^q$ (†) for every $n\in\mathbb{Z}$. Note that for $d<0$ we run into a problem as for fixed n the sequence $x_{n+d}=x_n^{1/q}, x_{n+2d}=x_n^{1/q^2},\dots$ could not tend to zero and so $\sum[x_n]\pi^n$ would not converge in B (a shadow of the fact to come that $B^{\varphi=\pi^d}=0\ \forall d<0$).
But for $d>0$ then any tuple $(x_0,\dots,x_{d-1})\in m_F^d$ gives a well-defined element of $B^{\varphi=\pi^d}$ as above via (†).

Rmk 13: It was asked during the talk why the def^n of P, and so of X, does not depend on the choice of uniformiser π. In their book, Fargues & Fontaine explain the following:
•) Suppose that π' is another uniformiser of $\mathcal{O}_E$. By Hilbert 90, $\exists\, u\in W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)^\times\subseteq A_{inf}^\times$ s.t

$$\frac{\varphi(u)}{u}=\frac{\pi'}{\pi}.$$

•) There is an iso $B^{\varphi=\pi^d} \longrightarrow B^{\varphi=\pi'^d}$, $f \mapsto u^d f$

Taking direct sum over all these defines an iso of graded rings $P_\pi \to P_{\pi'}$,

$\Rightarrow X = \mathrm{Proj}(P)$ doesn't depend on choice of π.

Rmk 14: It was pointed out after the talk by Andreas that a more naive 1-dim^l heuristic picture is to view a fund. domain for φ-action on $|Y|$ as a small closed interval, and identifying the boundary gives a circle, i.e. a $\mathbb{P}^1$.

§ "Appendix"

We just quickly explain the following fact, which we implicitly applied to $(B^b, \|-\|_I)$ in §2.

Lemma: Suppose A is a ring equipped with a norm $\|-\|$ satisfying $\|ab\| \leq \|a\| \cdot \|b\|$. Assume that $\exists \pi \in A^\times$ s.t. $0 < \|\pi\| < 1$ and $\|\pi^{-n}\| \to \infty$ as $n \to \infty$. Then a sequence $(a_n)_{n\geq 0}$ in $A^\circ = \{a \in A \mid \|a\| \leq 1\}$ is Cauchy w.r.t. $\|-\|$ if and only if it is Cauchy for the π-adic topology. Consequently, the completion $\hat{A}$ w.r.t $\|-\|$ is canonically isomorphic to $\widehat{A^\circ}^\pi[\frac{1}{\pi}]$, where $\widehat{A^\circ}^\pi :=$ π-adic completion.

Proof: Since $\|\pi\| < 1$ and $\|ab\| \leq \|a\| \cdot \|b\|$, it follows that $\|\pi^n\| \to 0$ as $n \to \infty$ and thus that if a sequence $(a_n)_{n \geq 0}$ is Cauchy w.r.t π-adic topology then it is Cauchy w.r.t $\|-\|$.
Conversely, choose $\varepsilon > 0$ and $N \gg 0$ s.t. $\|f_n - f_N\| \leq \varepsilon$ for all $n \geq N$. Furthermore, using that $\|\pi^{-m}\| \to \infty$, choose M_ε maximal s.t. $\|\pi^{-M_\varepsilon}\| \leq \frac{1}{\varepsilon}$. Then one has

$$\|(f_n - f_N)\pi^{-M}\| \leq \|f_n - f_N\| \cdot \|\pi^{-M_\varepsilon}\|$$
$$\leq \varepsilon \cdot \frac{1}{\varepsilon} = 1$$

$\Rightarrow (f_n - f_N)\pi^{-M_\varepsilon} \in A^0$ and hence
$f_n - f_N \in \pi^{M_\varepsilon} A^0$.
Noting that $M_\varepsilon \to \infty$ as $\varepsilon \to 0$, this shows that (f_n) is π-adically Cauchy. $\square$