

Newton Polygons I

26.11.25

Want to introduce the Newton Polygons of elements in A_{inf}

We will introduce them in greater Generality.

K non archimedean field with val. $v: K \rightarrow \mathbb{R} \cup \{\infty\}$
convention $v(0) = +\infty$

Def 1: Let $f(T) = \sum_{i=0}^n a_i T^i \in K[T]$.

We define the Newton Polygon $\text{New}_{poly}(f)$

as the largest convex function lying below the set of points $\{(i, v(a_i))\}_{i=0}^n$.

Prop 2: Let $x_0, \dots, x_n \in \bar{K}$ be the zeros of f . Then

$-v(x_0), \dots, -v(x_n)$, are exactly the slopes of $\text{New}_{poly}(f)$ with correct multiplicity.

(each slope repeated exactly as many times as the multiplicity of the corresponding zero)

Example 3

$$(1) f(T) = T - p^2$$

$$\text{Hence: } (0, v(p^2)) = (0, 2)$$

$$(1, v(1)) = (1, 0)$$



Slope = -2

Thus, $v(\alpha) = 2$ and
there is 1 root with valuation 2
for f .

Set: $K = \mathbb{Q}_p$,

For $x \neq 0$ in $\mathbb{Q}$, write $x = p^k \frac{m}{n}$
 $m, n \in \mathbb{Z}$ not divisible by p .

Then define

$$v: \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{\infty\}$$

$$x \mapsto v(x) = k,$$

$$v(0) = +\infty$$

Satisfying: $v(xy) = v(x) + v(y)$

$$v(x+y) \geq \min\{v(x), v(y)\}$$

$$v(1) = 0$$

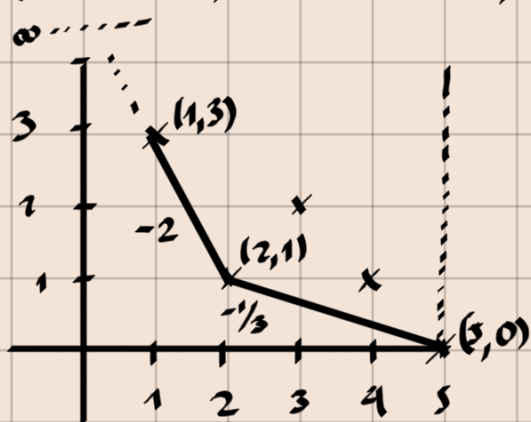
$$v(p^k) = k$$

$$v(-p^k) = k$$

$$(2) \quad \Phi(T) = p^3 T + p T^2 + p^2 T^3 - p T^4 + T^5$$

The $\text{New}_{\text{reg}}(\Phi)$ lies under the set of points:

$$\{(0, \infty), (1, 3), (2, 1), (3, 2), (4, 1), (5, 0)\}$$



$$v(\alpha) = 2$$

$$v(\beta) = \frac{1}{3}$$

Have:

$$1 \text{ root of val } 2 \Leftarrow \text{ if } \lambda_1 = 2$$

$$3 \text{ roots of val } \frac{1}{3} \Leftarrow \text{ if } \lambda_2 = \frac{1}{3}$$

$$0 \text{ is a root} \Leftarrow \text{ if } \lambda_0 = \infty$$

$$\text{Set } \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \text{ and } \mathcal{F} = \{\varphi: \mathbb{R} \rightarrow \bar{\mathbb{R}}\}$$

Def 4 We define:

• Legendre transform

$$\begin{aligned} \mathcal{L}: \mathcal{F} &\longrightarrow \mathcal{F} \\ \varphi &\longmapsto (\lambda \longmapsto \overbrace{\inf_{x \in \mathbb{R}} \{ \varphi(x) + \lambda x \}}^{\mathcal{L}(\varphi)(\lambda)}) \end{aligned}$$

• Inverse Legendre transform

$$\begin{aligned} \tilde{\mathcal{L}}: \mathcal{F} &\longrightarrow \mathcal{F} \\ \psi &\longmapsto (x \longmapsto \overbrace{\sup_{\lambda \in \mathbb{R}} \{ \psi(\lambda) - \lambda x \}}^{\tilde{\mathcal{L}}(\psi)(x)}) \end{aligned}$$

Think: Legendre transform interchange x -coordinates with slopes

Prop 5
$$\begin{aligned} \tilde{\mathcal{L}}(\varphi)(x) &= \sup_{\lambda \in \mathbb{R}} \{ \varphi(\lambda) - \lambda x \} = - \inf_{\lambda \in \mathbb{R}} \{ -\varphi(\lambda) + \lambda x \} = -\mathcal{L}(-\varphi)(x) \\ &\Rightarrow \tilde{\mathcal{L}}(\varphi) = -\mathcal{L}(-\varphi) \end{aligned}$$

Example 6: consider $Q(x) = ax + b$, $a, b \in \mathbb{R}$

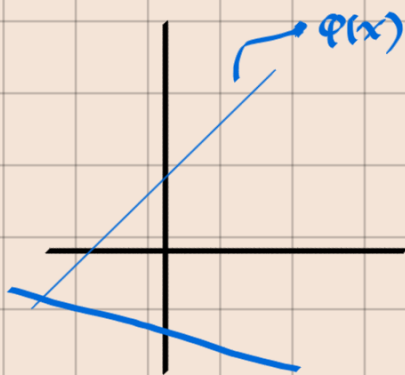
Example 6 $Q(x) = ax + b$

By definition of L by $\sup$ maximum of the minimum
 $L(Q)(\lambda) = \inf_{x \in \mathbb{R}} \{Q(x) + \lambda x\}$ largest c which satisfy the inequality (*)

$$\Rightarrow L(Q)(\lambda) \leq Q(x) + \lambda x, \forall x \in \mathbb{R}$$

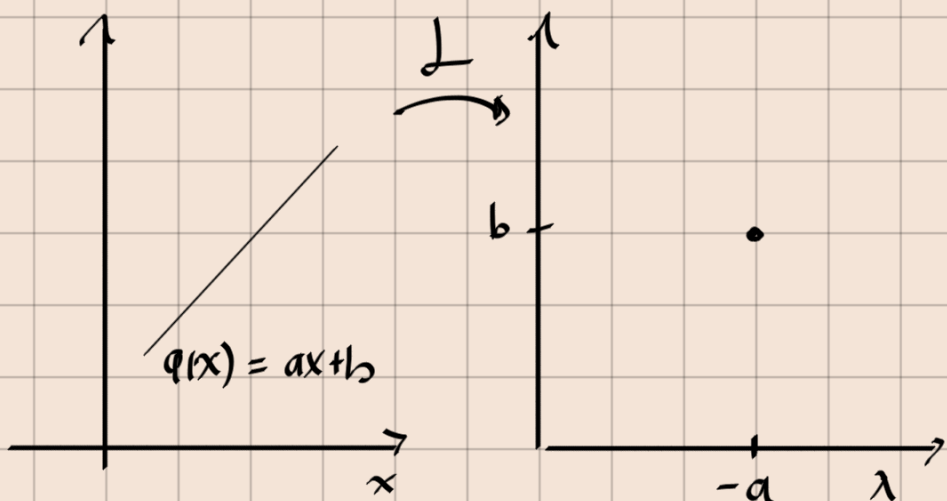
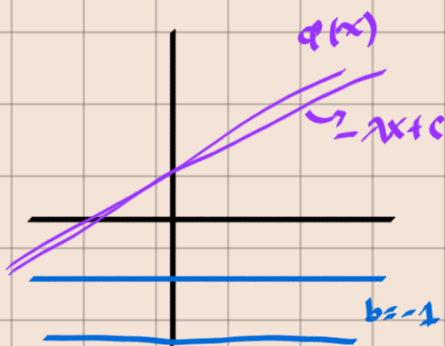
$$\Leftrightarrow \underbrace{L(Q)(\lambda) - \lambda x}_{=: c} \leq Q(x) \quad (*)$$

$$\Rightarrow -\lambda x + c \leq ax + b \quad \forall x \in \mathbb{R} \Leftrightarrow \overbrace{x(-\lambda - a) + c - b}^L \leq 0 \quad \forall x \in \mathbb{R}$$



either $c = -\infty$ or
 L is a bounded line then
 $\lambda = -a \Rightarrow c \leq b$ but since
 we want c to be the
 max of the by def. then
 $c = b$.

Also: $\tilde{L} L(Q) = Q$



Computations:

$$L(q)(\lambda) = \inf_{x \in \mathbb{R}} \{q(x) + \lambda x\} = \inf_{x \in \mathbb{R}} \{ax + b + \lambda x\}$$

$$= b + \inf_{x \in \mathbb{R}} \{x(a + \lambda)\}$$

- If $a + \lambda = 0 \Rightarrow \inf_{x \in \mathbb{R}} \{x(a + \lambda)\} = 0$
- If $a + \lambda > 0 \Rightarrow L(q)(\lambda) = b$ when $\lambda = -a$.

FALTA 

Conclusion: The height where the supporting lines touch the vertical axis (its intercept) is $L(q)(\lambda)$

- The slope of the line (which is $-\lambda$) becomes the x -coordinate in the new "dual" picture.

Def 7 Let $\varphi \in F$ and $x \in \mathbb{R}$.

φ admits supporting (resp. capping) line at x of slope λ if

- $\varphi(x) \neq \pm\infty$ and
- $\varphi(y) \geq \varphi(x) + \lambda(y-x)$

(resp) $\varphi(y) \leq \varphi(x) + \lambda(y-x) \quad \forall y \in \mathbb{R}$.

If $\varphi(x) = \pm\infty$, then we call each linear function
 $c + \lambda(y-x)$, $c, \lambda \in \mathbb{R}$

a supporting line if $\varphi(y) \geq c + \lambda(y-x)$
capping line if $\varphi(y) \leq c + \lambda(y-x)$.

Def 8: For all $x, y \in \mathbb{R}$, $t \in [0, 1]$, a function $\varphi: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is called

convex: if $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$

concave: if $\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y)$

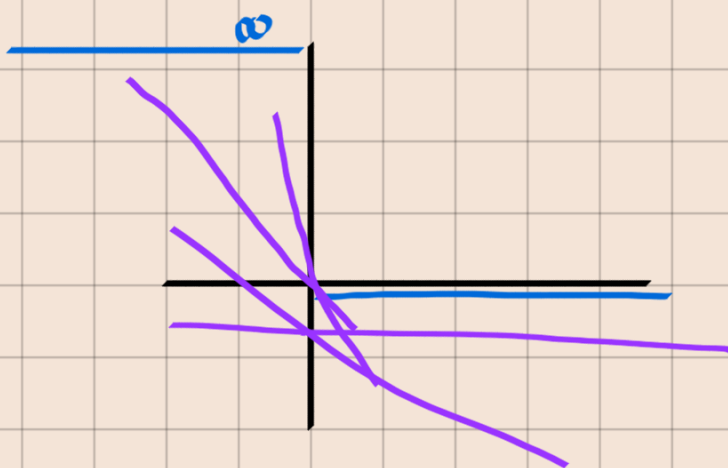
Prop: Thus a function $\varphi \in F$ is convex (resp. concave) iff it admits a supporting (resp. capping) line at each $x \in \mathbb{R}$.

Def 9: We call a convex (resp. concave) function $\varphi \in F$ non-extendible if φ is the supremum over all its supporting lines (resp. the infimum over all its capping lines).

Example

The function

$$\varphi(x) = \begin{cases} \infty & x \leq 0 \\ 0 & x > 0 \end{cases}$$



is convex and extendable

$$\varphi(tx + (1-t)y) \leq +\infty$$

$$\tilde{\varphi}(x) = \begin{cases} \infty, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

Note that $L(\varphi) = L(\tilde{\varphi})$ is the function $\lambda \mapsto \begin{cases} -\infty, & \lambda < 0 \\ 0, & \lambda \geq 0. \end{cases}$

Def 10: Let $\varphi \in \mathcal{F}$. Then, the non-extendable convex (resp concave) hull below (resp above) of φ is defined as the supremum over all its supporting lines (resp. the inf over all its capping lines).

Prop 11: Let $\varphi, \psi \in \mathcal{F}$.

(1) $L(\varphi)$ is non-ext. concave, $\tilde{L}(\varphi)$ is non-ext. convex.

(2) If $\varphi \leq \psi$, then $L(\varphi) \leq L(\psi)$ and $\tilde{L}(\varphi) \leq \tilde{L}(\psi)$.

(3) $\tilde{L}L(\varphi) \leq \varphi$ and $\varphi \leq L\tilde{L}(\varphi)$.

(4) If $\varphi(x) \neq \pm\infty$ admits a supporting line at x of slope λ then $L(\varphi)$ admits a capping line at $-\lambda$ of slope x .

(5) $\tilde{L}L(\varphi)$ is the non-extendable convex function below φ .

(6) $L, \tilde{L}$ define inverse bijections

$$L: \text{Conv}^{\text{ne}} \rightarrow \text{Conc}^{\text{ne}}, \quad \tilde{L}: \text{Conc}^{\text{ne}} \rightarrow \text{Conv}^{\text{ne}}$$

Proof:

(1) Recall $L(\varphi) := \inf_{x \in \mathbb{R}} \{ \varphi(x) + \lambda x \}$

For each fixed x define

$$l_x: \mathbb{R} \rightarrow \mathbb{R}$$

$$\lambda \mapsto l_x(\lambda) := \varphi(x) + \lambda x, \quad x \in \mathbb{R}$$

Moreover $L(\varphi)(\lambda) = \inf_{x \in \mathbb{R}} l_x(\lambda)$

So, $L(\varphi)$ is a family of functions that all lie $\geq L(\varphi)$, i.e. they are capping lines of $L(\varphi)$.

Thus, $L(\varphi) = \inf \{ l(\lambda) \mid l \text{ is capping line of } L(\varphi) \}$.

This implies the statement for $\tilde{L}$ using $\tilde{L}(\varphi) = -L(-\varphi)$ or just using the definition.

$$\tilde{L}(\gamma)(x) := \sup_{\lambda \in \mathbb{R}} \{ \gamma(\lambda) - \lambda x \}$$

Defining $h_x(\lambda) := \gamma(\lambda) - \lambda x$ we see that $\tilde{L}(\gamma)$ is the supremum of all functions in x that lie $\leq \tilde{L}(\gamma)$ i.e. supporting lines of $\tilde{L}(\gamma)$.

(2) Follows directly from the definition:

Take $x \in \mathbb{R}$.

$$If \quad \varphi(x) \leq \psi(x) \Leftrightarrow \varphi(x) + \lambda x \leq \psi(x) + \lambda x$$

$$\Leftrightarrow \inf_{x \in \mathbb{R}} \{ \varphi(x) + \lambda x \} \leq \inf_{x \in \mathbb{R}} \{ \psi(x) + \lambda x \}$$

$$\Leftrightarrow L(\varphi) \leq L(\psi).$$

Similar for $\tilde{L}(\varphi) \leq \tilde{L}(\psi)$.

(3) Recall $L(\varphi)(\lambda) = \inf_{x \in \mathbb{R}} \{ \varphi(x) + \lambda x \}$

$$\tilde{L}(\psi)(x) = \sup_{\lambda \in \mathbb{R}} \{ \psi(\lambda) + \lambda x \}$$

$$\bullet \quad \tilde{L}L(\varphi) = \sup_{\lambda \in \mathbb{R}} \{ \inf_{y \in \mathbb{R}} \{ \varphi(y) + \lambda y \} - \lambda x \}$$

For any point $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\inf_{y \in \mathbb{R}} \{ \varphi(y) + \lambda y \} \leq \varphi(x) + \lambda x$

$$\Leftrightarrow \inf_{y \in \mathbb{R}} \{ \varphi(y) + \lambda y \} - \lambda x \leq \varphi(x) + \lambda x - \lambda x$$

$$\Leftrightarrow \inf_{y \in \mathbb{R}} \{ \varphi(y) + \lambda y \} - \lambda x \leq \varphi(x)$$

Thus $\sup_{\lambda \in \mathbb{R}} \{ \inf_{y \in \mathbb{R}} \{ \varphi(y) + \lambda y \} - \lambda x \} \leq \sup_{\lambda \in \mathbb{R}} \varphi(x) = \varphi(x)$

$$\bullet \quad L\tilde{L}(\varphi) = \inf_{x \in \mathbb{R}} \{ \sup_{\delta \in \mathbb{R}} \{ \varphi(\delta) - \delta y \} + \lambda x \}$$

For any $\lambda \in \mathbb{R}$:

$$\sup_{\delta \in \mathbb{R}} \{ \varphi(\delta) - \delta y \} \geq \varphi(\lambda) - \lambda x, \quad x, y \in \mathbb{R}$$

$$\sup_{\delta \in \mathbb{R}} \{ \varphi(\delta) - \delta y \} + \lambda x \geq \varphi(\lambda) - \lambda x + \lambda x$$

$$\sup_{\delta \in \mathbb{R}} \{ \varphi(\delta) - \delta y \} + \lambda x \geq \varphi(\lambda)$$

$$\inf_{x \in \mathbb{R}} \{ \sup_{\delta \in \mathbb{R}} \{ \varphi(\delta) - \delta y \} + \lambda x \} \geq \varphi(\lambda).$$

(A) wts. $L(\varphi)$ has a capping line at $-\lambda$ of slope x ,
i.e.

Recall def capp line: A function γ admits a capping line at u of slope m iff the straight line through the point $(u, \gamma(u))$ with slope m lies above the whole graph, i.e.

$$\gamma(v) \leq \gamma(u) + m(v-u), \quad \forall v \in \mathbb{R}$$

Now take $\gamma = L(\varphi)$, $u = -\lambda$ and $m = x$, to get

$$L(\varphi)(\mu) \leq L(\varphi)(-\lambda) + x(\mu + \lambda), \quad \forall \mu \in \mathbb{R}$$

The line through $(-\lambda, L(\varphi)(-\lambda))$ with slope x , caps $L(\varphi)$.

Start the proof:

$$\text{By def } L(\varphi)(\mu) = \inf_{y \in \mathbb{R}} \{ \varphi(y) + \mu y \} \leq \varphi(x) + \mu x \quad (*)$$

Since $\varphi(x)$ has a supporting line at x w/ slope λ , then

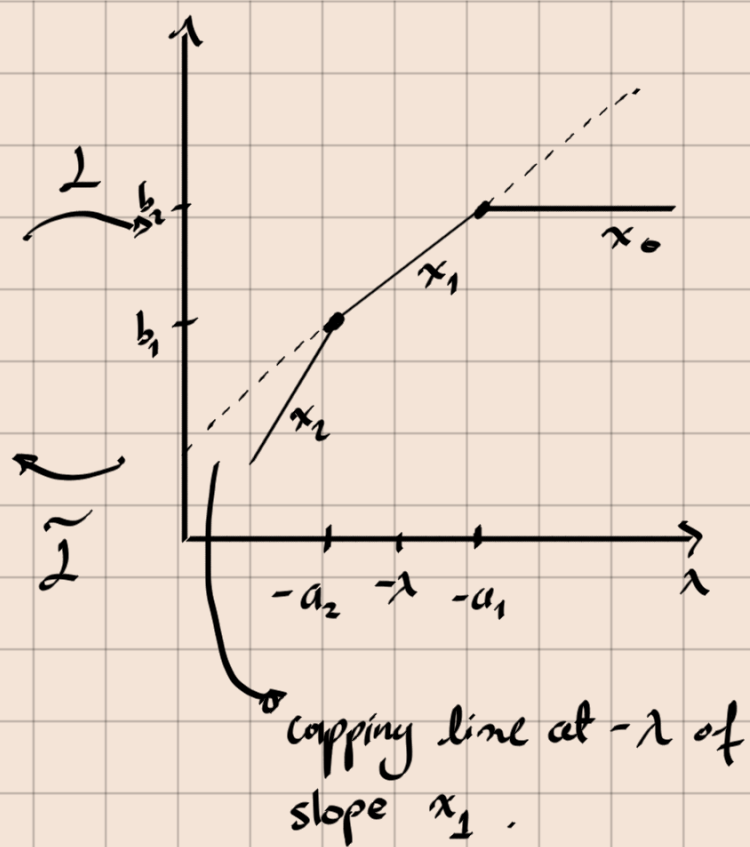
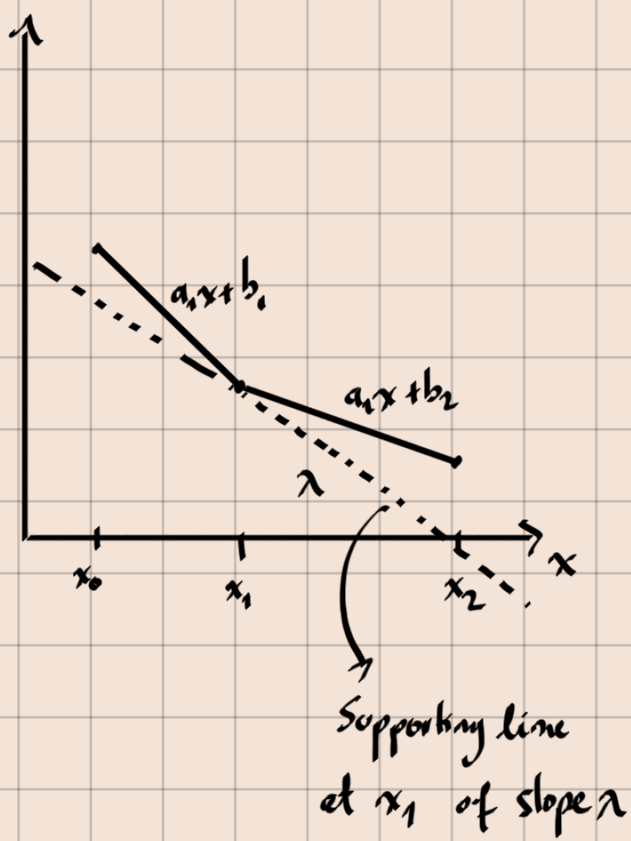
$$\varphi(y) \geq \varphi(x) + \lambda(y-x) \quad \forall y \in \mathbb{R}$$

$$\Leftrightarrow \varphi(y) - \lambda y \geq \varphi(x) - \lambda x$$

$$\Rightarrow L(\varphi)(-\lambda) + x(\lambda + \mu) \geq \varphi(x) - \lambda x + x(\lambda + \mu)$$

$$\Rightarrow L(\varphi)(-\lambda) + x(\lambda + \mu) \geq \varphi(x) + x\mu \geq L(\varphi)(\mu).$$

Notice that if $\varphi(x) = \pm\infty$, also holds.



- (5). By (1) and (3) we have:
- $\tilde{J}(Q) \in \text{Conv}^{ne}$
 - $\tilde{J}(Q) \leq Q$

There are many convex small functions below Q .

The Newton polygon, however, is the largest convex function below: The convex hull.

So, we still need to prove:

$\tilde{J}(Q)$ is the largest convex function $\leq Q$.

Equivalently: $\tilde{J}(Q) = \sup \{ l \text{ suppo. lines of } Q \}$
(non-extendable convex hull below Q).

W.T.S. $\tilde{J}(Q)$ lies above every supporting line of Q .

i.e. we need to prove that for every supporting line l of Q we have $l \leq \tilde{J}(Q)$.

By Example 6 we have that for $l(x) = ax+b$, $\tilde{J}(l) = l$.

Let l be any supporting line of Q . Then $l \leq Q$.

By (2) we have

$$l \leq Q \rightarrow J(l) \leq J(Q)$$

$$\rightarrow \tilde{J}(l) \leq \tilde{J}(Q)$$

$$\rightarrow l \leq \tilde{J}(Q).$$

(6) Claim: $\mathcal{I}(\varphi) \geq \psi \iff \varphi \geq \tilde{\mathcal{I}}(\psi)$

Proof:

$$\bullet \mathcal{I}(\varphi) \geq \psi \xRightarrow{\tilde{\mathcal{I}}(2)} \tilde{\mathcal{I}} \mathcal{I}(\varphi) \geq \tilde{\mathcal{I}}(\psi) \quad \begin{matrix} \parallel \\ \varphi \end{matrix}$$

$$\bullet \varphi \geq \tilde{\mathcal{I}}(\psi) \xRightarrow{1} \mathcal{I}(\varphi) \geq \mathcal{I} \tilde{\mathcal{I}}(\psi) \geq \psi \quad (3)$$

$$\mathcal{H}_{\mathcal{F}}(\mathcal{I}(\varphi), \psi) = \begin{cases} \{*\} & \mathcal{I}(\varphi) \geq \psi \\ \emptyset & \text{else} \end{cases}$$

$$\mathcal{H}_{\mathcal{F}}(\varphi, \tilde{\mathcal{I}}(\psi)) = \begin{cases} \{*\} & \varphi \geq \tilde{\mathcal{I}}(\psi) \\ \emptyset & \text{else} \end{cases}$$

$$\iff \mathcal{H}_{\mathcal{F}}(\mathcal{I}(\varphi), \psi) \cong \mathcal{H}_{\mathcal{F}}(\varphi, \tilde{\mathcal{I}}(\psi))$$

Thus, (2) + (3)

$$\mathcal{I} \dashv \tilde{\mathcal{I}} : \mathcal{F}(\leq) \longrightarrow (\mathcal{F}, \leq)$$

Now: Adjunction implies an equivalence on the fixed points:

+ (1) + (5):

Which fixed points are we talking about?

- Non-extendable convex functions = fixed points of the convex hull operator

$$\mathcal{T} := \tilde{\mathcal{I}} \mathcal{I}, \quad \mathcal{T}(\varphi) = \tilde{\mathcal{I}} \mathcal{I}(\varphi) \quad \text{i.e.}$$

$$\varphi \text{ non-extendable convex func.} \iff \tilde{\mathcal{I}} \mathcal{I}(\varphi) = \varphi$$

- Non-extendable concave functions = fixed points of upper hull

$$\mathcal{S} := \mathcal{I} \tilde{\mathcal{I}}, \quad \mathcal{S}(\psi) = \mathcal{I} \tilde{\mathcal{I}}(\psi)$$

$$\psi \text{ non-extn concave func.} \iff \mathcal{I} \tilde{\mathcal{I}}(\psi) = \psi$$

So:

$$\bullet \text{Conv}^{\text{ne}} = \{\varphi \mid \tilde{\mathcal{I}} \mathcal{I}(\varphi) = \varphi\}$$

$$\bullet \text{Conc}^{\text{ne}} = \{\psi \mid \mathcal{I} \tilde{\mathcal{I}}(\psi) = \psi\}$$

Thus

$$\tilde{\mathcal{I}} \circ \mathcal{I} = \text{id}_{\text{Conv}^{\text{ne}}}$$

$$\mathcal{I} \circ \tilde{\mathcal{I}} = \text{id}_{\text{Conc}^{\text{ne}}}$$

w.t.s. $L: \text{Conv}^{\text{ne}} \rightarrow \text{Conc}^{\text{ne}}$, $\tilde{L}: \text{Conc}^{\text{ne}} \rightarrow \text{Conv}^{\text{ne}}$
are inverse bijections.

(1) L sends fixed points of $\tilde{L}L$ to fixed points of $L\tilde{L}$ and dually for $\tilde{L}$.

(2) On those fixed points, L and $\tilde{L}$ undo each other.

Proof

L and $\tilde{L}$ send fixed points to fixed points

- Start with non-extendable convex q :

$$\text{Assume } \tilde{L}L(q) = q \iff \underset{\text{"}}{L\tilde{L}L(q)} = L(q) \\ S(L(q))$$

Thus $L(q)$ is a fixed point of $S = L\tilde{L}$ i.e. non extendable concave. So

$$L: \text{Conv}^{\text{ne}} \longrightarrow \text{Conc}^{\text{ne}}$$

- Start with non-extendable concave ψ .

$$\text{Assume } L\tilde{L}(\psi) = \psi \iff \underset{\text{"}}{\tilde{L}L\tilde{L}(\psi)} = \tilde{L}(\psi) \\ T(\tilde{L}(\psi))$$

Thus $\tilde{L}(\psi)$ is a fixed point of $T = \tilde{L}L$, i.e. non extendable convex. So

$$\tilde{L}: \text{Conc}^{\text{ne}} \longrightarrow \text{Conv}^{\text{ne}}$$

$L, \tilde{L}$ are mutual inverses:

$$\tilde{L} \circ L = \text{id}_{\text{convex}} \quad L \circ \tilde{L} = \text{id}_{\text{concave}}.$$

Summary:

- From (2)+(3) we know $L \dashv \tilde{L}$ so we get a convex hull operator $T = \tilde{L}L$ and a concave hull operator $S = L\tilde{L}$.
- By (1)+(5),
fixed points of T are exactly non-extendable convex funct.
fixed points of S are exactly non-extendable concave funct.
- If $\tilde{L}L(q) = q$ Then $L(q)$ satisfies

$$L\tilde{L}(L(q)) = L(\tilde{L}L(q)) = L(q)$$

so $L(q)$ is non-extendable concave.

Dually: $\tilde{L}$ sends non-extendable concave to non-extend convex.

- Finally:

$$\tilde{L}L(q) = q \quad \text{and} \quad L\tilde{L}(\gamma) = \gamma$$

So L and $\tilde{L}$ restrict to inverse bijections between these fixed points.

Thus, (2) + (3) imply

$$(\bar{F}, \leq) \begin{matrix} \xrightarrow{L} \\ \xleftarrow{\tilde{L}} \end{matrix} (\bar{F}, \leq) \text{ are adjoint functors}$$

and this implies an equivalence on the fixed points

$$\bigcup_{\text{Fix}}(\bar{L}, \tilde{L})$$

$$\bigcup_{\text{Fix}}(\tilde{L}, \bar{L})$$

$$\left[\begin{array}{l} \text{Fix } \bar{L} \subset \bar{F} = \{ \varphi : \varphi \text{ s.t. } L(\varphi) = \varphi \\ \text{Mor: } \varphi_1 \rightarrow \varphi_2 \Leftrightarrow \varphi_1 \geq \varphi_2 \end{array} \right]$$

$$\tilde{L} L(\varphi) = \varphi$$

$$L \tilde{L}(\tau) = \tau$$

Includ:

$$\varphi$$

$$\longmapsto$$

$$L(\varphi) = \tau$$

$$\left[\begin{array}{l} L(\tilde{L} L(\varphi)) = L(\varphi) = \tau \\ \varphi \end{array} \right]$$

$$\tilde{L}(\tau) = \varphi$$

$$\longleftarrow$$

$$\tau$$

$$\tilde{L} \underbrace{L \tilde{L}(\tau)}_{\tau} = \tilde{L}(\tau) = \varphi$$

Thus by (1) + (5),

$$L: \text{Con}^{\text{ne}} \rightarrow \text{Con}^{\text{ne}}, \quad \tilde{L}: \text{Con}^{\text{ne}} \rightarrow \text{Con}^{\text{ne}}$$

are mutual inverses.

Remark 11.1 Coming back to non-archimedean field K with valuation $v: K \rightarrow \mathbb{R} \cup \{\infty\}$.

Choose a polynomial $f(T) = \sum_{i=0}^n a_i T^i \in K[T]$.

By definition $\text{Newpoly}(f)$ is the largest convex function below the set $\{(i, v(a_i))\}_{i \in \mathbb{Z}}$.

One can show that prop 11 implies that

$$L(\text{Newpoly}(f))(r) = \inf_{i \in \mathbb{Z}} \{v(a_i) + ri\} =: v_r(f)$$

Proof: Denote $\varphi := \text{Newpoly}(f)$.

$$\text{By def: } L(\varphi)(r) = \inf_{x \in \mathbb{R}} \{ \varphi(x) + rx \}$$

$$" \leq " \quad L(\varphi)(r) \leq \inf_{i \in \mathbb{Z}} \{v(a_i) + ri\} :$$

For any $i \in \mathbb{Z}$ we have

$$\varphi(i) \leq v(a_i) \text{ because } \varphi \text{ lies below each point}$$

$$\Rightarrow L(\varphi)(r) = \inf_{x \in \mathbb{R}} \{ \varphi(x) + rx \} \leq \varphi(i) + ri \leq v(a_i) + ri$$

Taking inf over i

$$\Rightarrow L(\varphi)(r) \leq \inf_{i \in \mathbb{Z}} \{v(a_i) + ri\}$$

$$" \geq " \quad L(\varphi)(r) \geq \inf_{i \in \mathbb{Z}} \{v(a_i) + ri\} :$$

$$\text{Fix an } r \text{ and set } c := \inf_{i \in \mathbb{Z}} \{v(a_i) + ri\}$$

Consider the line of slope $-r$: $l_r(x) = -rx + c$.

By def of c , we have

$$l_r(i) = -ri + c \leq v(a_i)$$

So l_r lies below every point $(i, v(a_i))$.

Recall that Q is the largest convex function below all those points. Since l_r is convex and lies below all points, it must lie below Q and by Prop 1. (5) we have:

$$l_r(x) \leq Q(x) \quad \forall x$$

$$\Leftrightarrow -rx + c \leq Q(x)$$

$$\Leftrightarrow c \leq Q(x) + rx$$

$$\Rightarrow c \leq \inf_{x \in \mathbb{R}} \{Q(x) + rx\} = L(Q)(r)$$

$$\Rightarrow \inf_{i \in \mathbb{Z}} \{v(a_i) + ir\} \leq L(Q)(r)$$

Thus

$$L(Q)(r) = \inf_{i \in \mathbb{Z}} \{v(a_i) + ir\}.$$

Important: The functions v are not only norms, but valuations
i.e. multiplicative norms.

Lemma 13. Let $v \in \mathbb{R}$ and $f, g \in K[T]$. Then,

$$v_r(f \cdot g) = v_r(f) + v_r(g).$$

Proof: Recall that $v_r(h) = \inf_{\substack{v \in \mathbb{R} \\ \lambda \in \mathbb{Z}}} \{v(c_\lambda) + r\lambda\}$

where $h = \sum_{\lambda=0}^{\infty} c_\lambda T^\lambda \in K[T]$.

Recall

Fact: Every polynomial over a field splits into linear factors over some extension. (Think in $K \subseteq \bar{K}$).

Then:

- Take an extension field $L \supset K$ in which f splits (e.g. a splitting field or $\bar{K}$).
- Extend the valuation from K to L :

$$v: \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\},$$

$$L/K \text{ finite} \Rightarrow \exists! \quad w: L \rightarrow \mathbb{R} \cup \{\infty\} \\ \text{valuation s.t. } w|_K = v.$$

- Now, in $L(T)$ we can write

$$f(T) = c \prod_{j=1}^m (T - a_j), \quad a_j \in L.$$

Notice: The coefficients are the same whether we view f as a polynomial over K or over any extension L .

So, if we prove the equality $v_r(fg) = v_r(f) + v_r(g)$

inside $L[T]$, it is automatically the same equality in $K[T]$.

"Pass to the field where we can factor f , prove the statement there and then come back"

Notice: It is enough to prove the statement for $f = T - a$:

Assume we have proved the following:

For every a and every polynomial h ,

$$v_r((T-a)h) = v_r(T-a) + v_r(h) \quad \forall h. \quad (*)$$

After extending K , we can write $f(T) = c \prod_{j=1}^m (T - a_j)$.

(W.T.S: For every polynomial f of the form $c \prod_{j=1}^m (T - a_j)$ and every $g \in K[T]$)
 $P(m) := v_r(fg) = v_r(f) + v_r(g)$.

Proof: Induction on $m \geq 0$ (degree f)

$P(m=0)$

• Case: $m=0 \Rightarrow f=c$. Then

$$v_r(fg) = v_r(c) + v_r(g) = v_r(f) + v_r(g)$$

because multiplying by a scalar just shifts all coefficient valuations by $v_r(c)$. So the lemma is true.

$f=c$ in K . Then $fg = cg$

$$\text{where } g(T) = \sum_{i=1}^n b_i T^i$$

$$\begin{aligned} \text{By def } v_r(cg) &= \inf \{ v_r(cb_i) + r_i \} \\ &= \inf \{ v_r(c) + v_r(b_i) + r_i \} \\ &= v_r(c) + \inf \{ v_r(b_i) + r_i \} \\ &= v_r(c) + v_r(g). \end{aligned}$$

$P(m=1)$

• Case: $m=1 \Rightarrow f = T - a$. Then

$$v_r((T-a)g) = v_r(T-a) + v_r(g)$$

Hypothesis index: assume the lemma holds for all polynomials

$P(m-1)$: with at most $m-1$ linear factors.
we want to show $P(m)$.

Define: $f = (T-a_m) \cdot h$, $h := c \prod_{j=1}^{m-1} (T-a_j)$,

First, we apply (*)

$$v_r(fg) = v_r((T-a_m)hg) = v_r(T-a_m) + v_r(hg) \quad (1)$$

Now, use the induction hypothesis on h (it has $m-1$ factors)

$$v_r(hg) = v_r(h) + v_r(g). \quad (2)$$

Plug (2) into (1)

$$v_r(fg) = v_r((T-a_m)hg) = v_r(T-a_m) + v_r(h) + v_r(g) \quad (3)$$

But again, by (*)

$$v_r(f) = v_r((T-a_m)h) = v_r(T-a_m) + v_r(h) \quad (4)$$

Plug (4) in (3)

$$v_r(fg) = v_r(f) + v_r(g).$$

So, by induction, the Lemma holds for all m i.e for all f .

We need to show (*)

$$v_r((T-a)h) = v_r(T-a) + v_r(h) \quad \forall h. \quad (*)$$

Notice that:

$$v_r(Tg) = r + v_r(g)$$

$$v_r(-ag) = v(a) + v_r(g).$$

Induct: $g = \sum_{i=0}^n b_i T^i$

$$\Rightarrow Tg = \sum_{i=0}^n b_i T^{i+1}$$

$$\Rightarrow v_r(Tg) = \inf \{v(b_i) + r(i+1)\}$$

$$= \inf \{v(b_i) + ri + r\}$$

$$= r + \inf \{v(b_i) + ri\}$$

$$= r + v_r(g).$$

Claim 1: $v_r((T-a)g) \geq v_r(T-a) + v_r(g)$

Claim 2: $v_r(f+g) \geq \min \{v_r(f), v_r(g)\}$

Proof 2:

Let $f = \sum_{i=0}^n a_i T^i$, $g = \sum_{i=0}^n b_i T^i$ and

$$f+g = \sum_{i=0}^n (a_i + b_i) T^i$$

$$v_r(f+g) = \inf_{i \in \mathbb{Z}_+} \{v(a_i + b_i) + ri\}$$

have:

$$v(a_i + b_i) + ri \geq \begin{cases} \inf v(b_i) + ri, & \text{if } \min \{v_r(f), v_r(g)\} = v_r(f) \\ \inf v(a_i) + ri, & \text{if } \min \{v_r(f), v_r(g)\} = v_r(g). \end{cases}$$

$$\geq \min \{v_r(f), v_r(g)\} \quad \forall i$$

$$\Rightarrow v_r(f+g) \geq \min \{v_r(f), v_r(g)\}.$$

Now, if $v_r(f) \neq v_r(g) \rightarrow v_r(f+g) = \min_{i \in \mathbb{Z}} \{v_r(f), v_r(g)\}$

Proof

WLOG Take $v_r(f) < v_r(g)$ (3*)

$$\rightarrow v_r(f+g) \geq v_r(f) \quad (4*)$$

Notice:

$$v_r(f) = v_r(f+g+(-g))$$

$$\geq \min \{ v_r(f+g), \overset{v_r(g)}{v_r(-g)} \} \quad \left\{ \begin{array}{l} \text{if } \min \{ v_r(f+g), v_r(g) \} = v_r(g) \\ \text{the } v_r(f) \geq v_r(g) \end{array} \right.$$

$$= v_r(f+g) \quad (5*)$$

$$\sqrt{v_r(-g) = \min \{ v(-b_i) + r_i \}} = v_r(g)$$

$$v(-b_i) = v(-1) + v(b_i) = v(b_i)$$

$$v(1) = v(1^2) = 2 v(-1) \Rightarrow v(-1) = 0$$

$$v(1) = v(1 \cdot 1) = v(1) + v(1)$$

$$\Rightarrow v(1) = 0$$

$$(4*) + (5*) \Rightarrow v_r(f) \geq v_r(f+g) \geq v_r(f) \Rightarrow "=".$$

Proof Claim 1: By claim 2 we have

$$v_r(Tg - ag) \geq \min \{ v_r(Tg), v_r(-ag) \}$$

$$\geq \min \{ r + v_r(g), v(a) + v_r(g) \}$$

$$= v_r(g) + v_r(T-a)$$

$$\overline{v_r(T-a)} = \inf \{ \overset{0}{v(t)} + r, \overset{v(a)}{v(-a)} + 0 \cdot r \}$$

$$= \inf \{ r, v(a) \}$$

$$\Rightarrow \text{claim 1} \quad \left[v_r((T-a)g) \geq v_r(g) + v_r(T-a) \right]$$

Notice that

$$v_r(Tg) \neq v_r(-ag) \Leftrightarrow r \neq v(a)$$

$$\begin{array}{cc} \parallel & \parallel \\ r + v_r(g) & v(a) + v_r(g) \end{array}$$

in part

$$r \neq v(a) \Rightarrow v_r(Tg - ag) = \min \{ r, v(a) \} + v_r(g)$$

$$v(t) + v(g) \rightarrow v(tg) = \min \{ v(t), v(g) \}$$

Claim 3 $F(r) = G(r)$ for $r \neq v(a)$.

Proof

Let $r < v(a) \Rightarrow G(r) = r$ and

$$v(Tg) = r + v(g) < v(a) + v(g) = v(Tg)$$

$$\Rightarrow v(Tg - ag) = \min\{v(Tg), v(-ag)\}$$

$$= v(Tg)$$

$$= r + v(g)$$

$$\Rightarrow v(Tg - ag) - v(g) = r \Rightarrow F(r) = r = G(r) \quad \forall r \in \mathbb{R}$$

Now, set

$$F(r) = v_r(g(Ta)) - v(g) \quad \text{and} \quad G(r) = \min\{r, v(a)\}$$

$$\Rightarrow F(r) \geq G(r) \quad \forall r \in \mathbb{R}$$

Similarly if $r > v(a)$ then

$$F(r) = v(a) = G(r) \quad \forall r \in \mathbb{R} \Rightarrow \text{"Claim 3"}$$

Claim 4 $F(r) = G(r)$ for $r = v(a)$

Proof:

For each fixed i , the function

$$r \mapsto v(c) + r \cdot i \quad \text{is an affine linear and}$$

therefore continuous function.

Fact: The minimum of finitely many continuous functions is continuous.

Hence $r \mapsto v_r(c_i)$ is continuous on $\mathbb{R}$.

Now:

$$v_{v(a)}(f+g) = \lim_{v(a) \neq r \rightarrow v(a)} v_r(f+g) = \lim_{r \rightarrow v(a)} (v_r(f) + v_r(g))$$

$$= \lim_{r \rightarrow v(a)} v_r(f) + \lim_{r \rightarrow v(a)} v_r(g)$$

$$= v_{v(a)}(f) + v_{v(a)}(g) \quad \square$$

Have that

$F: r \mapsto v_r(Tg - ag)$ cont. on $\mathbb{R}$

$G: r \mapsto \min\{r, v(a)\} + v_r(g)$ cont. on $\mathbb{R}$

$$\text{Now } F|_{\mathbb{R} \setminus \{v(a)\}} = G|_{\mathbb{R} \setminus \{v(a)\}}$$

Since $\mathbb{R} \setminus \{v(a)\}$ is dense in $\mathbb{R} \Rightarrow F = G$

$\square$

Remark

Let $h(T) = \sum_i a_i T^i$ and $\mathcal{Q}_h := \text{Newpoly}(h)$.

We have $L(\mathcal{Q}_h)(r) = \inf_i \{v(a_i) + r \cdot i\} =: v_r(h)$.

Lemma 13: $v_r(fg) = v_r(f) + v_r(g)$.

Then, for every $r \in \mathbb{R}$:

$$L(\mathcal{Q}_{fg})(r) = v_r(fg) = v_r(f) + v_r(g) = L(\mathcal{Q}_f)(r) + L(\mathcal{Q}_g)(r)$$

$$\text{Thus } L(\mathcal{Q}_{fg}) = L(\mathcal{Q}_f) + L(\mathcal{Q}_g).$$

Now, to analyze how the Newton Polygon of a product $f \cdot g$ can be described via $\text{Newpoly}(f)$ and $\text{Newpoly}(g)$, we need the convolution product of functions.

Def Let $\varphi, \psi \in \mathcal{T}$ s.t. $-\infty \notin \text{Im } \varphi \cup \text{Im } \psi$
(Thus sums like $\varphi(a) + \varphi(b)$ make sense in $\overline{\mathbb{R}}$)

The (infimal) convolution of φ and ψ is defined to

be the function

$$\begin{aligned} \varphi * \psi: \mathbb{R} &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto (\varphi * \psi)(x) = \inf_{\substack{a+b=x \\ a, b \in \mathbb{R}}} \{ \varphi(a) + \varphi(b) \}. \end{aligned}$$

The Legendre transform behaves well with convolution.

Lemma 15: Let $\varphi, \psi \in \tilde{\mathcal{F}}$ s.t. $-\infty \notin \text{Im } \varphi \cup \text{Im } \psi$.

(1) If φ, ψ are convex, then $\varphi * \psi$ is convex

(2) Under the Legendre transform, convolution becomes addition:

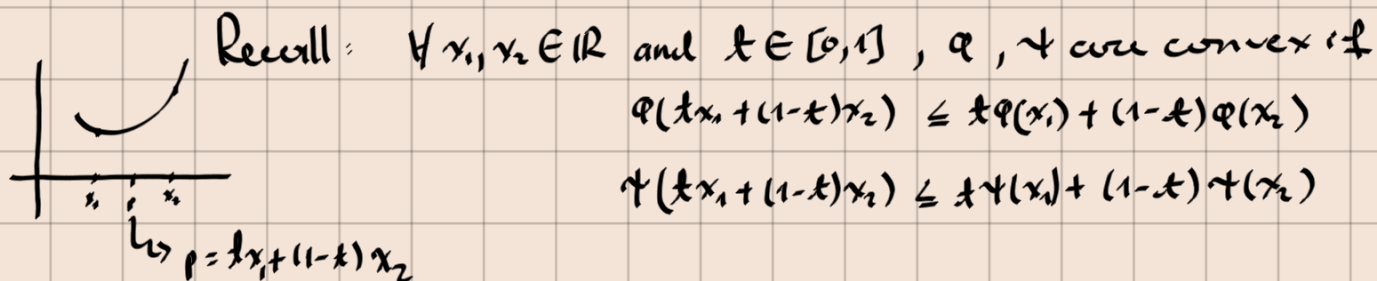
$$\mathcal{L}(\varphi * \psi) = \mathcal{L}(\varphi) + \mathcal{L}(\psi).$$

Proof:

(2) For $r \in \mathbb{R}$,

$$\begin{aligned} \mathcal{L}(\varphi * \psi)(r) &= \inf_{x \in \mathbb{R}} \{ (\varphi * \psi)(x) + rx \} \\ &= \inf_{x \in \mathbb{R}} \inf_{a+b=x} \{ \varphi(a) + \psi(b) + r(a+b) \} \\ &= \inf_{a,b \in \mathbb{R}} \{ \varphi(a) + \psi(b) + ra + rb \} \\ &= \inf_{a \in \mathbb{R}} \{ \varphi(a) + ra \} + \inf_{b \in \mathbb{R}} \{ \psi(b) + rb \} \\ &= \mathcal{L}(\varphi)(r) + \mathcal{L}(\psi)(r) \end{aligned}$$

(1) If φ, ψ are convex the $\varphi * \psi$ is convex



Now, wts that property but for $\varphi * \psi$, i.e

$$(\varphi * \psi)(tx_1 + (1-t)x_2) \leq t(\varphi * \psi)(x_1) + (1-t)(\varphi * \psi)(x_2)$$

Set $x = tx_1 + (1-t)x_2$.

wts

$$(\varphi * \psi)(x) \leq t(\varphi * \psi)(x_1) + (1-t)(\varphi * \psi)(x_2)$$

For each $i=1,2$ and for any $\varepsilon > 0$, choose a decomposition

$$x_i = a_i + b_i \quad \text{s.t.} \quad \varphi(a_i) + \psi(b_i) \leq (\varphi * \psi)(x_i) + \varepsilon. \quad (1)$$

Define $a := ta_1 + (1-t)a_2$, $b := tb_1 + (1-t)b_2$

Then

$$\begin{aligned} a+b &= \cancel{ta_1} + a_2 - \cancel{a_2t} + \cancel{tb_1} + b_2 - \cancel{tb_2} \\ &= t(a_1+b_1) + a_2(1-t) + b_2(1-t) \\ &= t(a_1+b_1) + (1-t)(a_2+b_2) \\ &= tx_1 + (1-t)x_2 \\ &= x \end{aligned}$$

So $a+b$ works!

Since φ and ψ are convex, we have

$$\Leftrightarrow \begin{aligned} \varphi(x a_1 + (1-x) a_2) &\leq x \varphi(a_1) + (1-x) \varphi(a_2) \\ \varphi(a) &\leq x \varphi(a_1) + (1-x) \varphi(a_2) \end{aligned} \quad \text{and}$$

$$\Leftrightarrow \begin{aligned} \psi(x b_1 + (1-x) b_2) &\leq x \psi(b_1) + (1-x) \psi(b_2) \\ \psi(b) &\leq x \psi(b_1) + (1-x) \psi(b_2) \end{aligned}$$

Now we compute:

$$\begin{aligned} (\varphi * \psi)(x) &= \inf_{u+v=x} \{ \varphi(u) + \psi(v) \} \\ &\leq \varphi(a) + \psi(b) \\ &\leq x \varphi(a_1) + (1-x) \varphi(a_2) + x \psi(b_1) + (1-x) \psi(b_2) \\ &= x (\varphi(a_1) + \psi(b_1)) + (1-x) (\varphi(a_2) + \psi(b_2)) \\ &\leq x ((\varphi * \psi)(x_1) + \varepsilon) + (1-x) ((\varphi * \psi)(x_2) + \varepsilon) \\ &= x (\varphi * \psi)(x_1) + (1-x) (\varphi * \psi)(x_2) + \varepsilon \end{aligned}$$

What we want holds when $\varepsilon \rightarrow 0$. Thus $(\varphi * \psi)$ is convex.



Corollary 16: For $f, g \in K[T]$

$$\text{New}_{\text{poly}}(f \cdot g) = \text{New}_{\text{poly}}(f) * \text{New}_{\text{poly}}(g)$$

Proof: Both sides are convex.

$$\begin{aligned} \tilde{L}(Q_{fg}) &= L(Q_f) + L(Q_g) \\ &= \tilde{L}(Q_f * Q_g) \end{aligned}$$

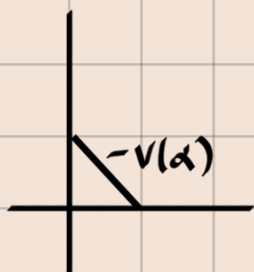
$$\Rightarrow Q_{fg} = Q_f * Q_g$$

$$\Rightarrow \text{New}_{\text{poly}}(f \cdot g) = \text{New}_{\text{poly}}(f) * \text{New}_{\text{poly}}(g)$$

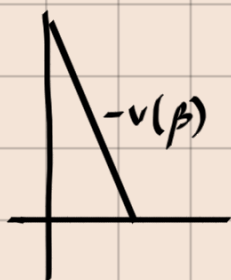
■

Example 17 : Take $f = T - \alpha$, $g = T - \beta$.

- $\text{New}_{\text{poly}}(f)$ is the largest convex polygon below $(0, v(\alpha))$
 $(1, 0)$



- $\text{New}_{\text{poly}}(g)$ is the largest convex polygon below $(0, v(\beta))$
 $(1, 0)$



By corollary 6.16 , the product

$$(T - \alpha)(T - \beta) = T^2 - T(\alpha + \beta) + \alpha\beta$$

has Newton Polygon of these two segment polygons.

Equivalently:

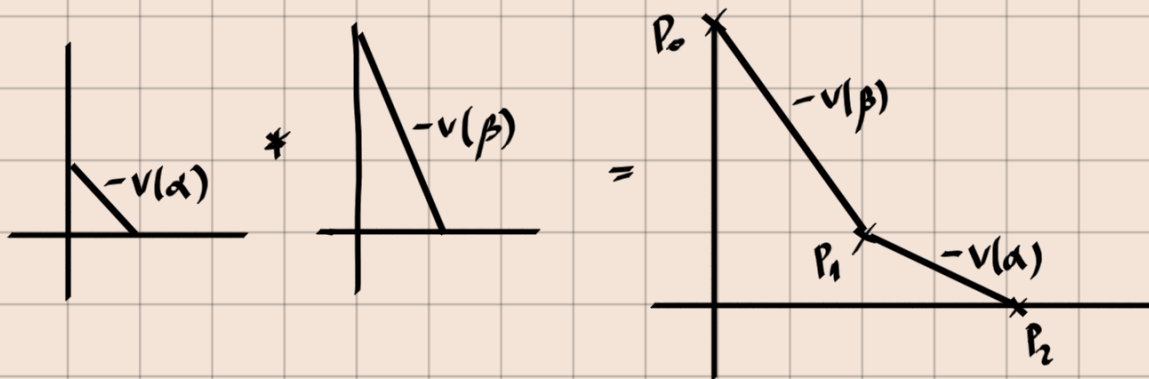
The polygon of $f \cdot g$ has two segments whose slopes are $-v(\alpha)$, $-v(\beta)$.

Notice that $\text{New}_{\text{poly}}(fg)$ is the largest convex polygon below the points:

$$P_0 = (0, v(\alpha\beta)) , P_1 = (1, v(\alpha + \beta)) , P_2 = (2, 0)$$

" $(0, v(\alpha) + v(\beta))$

Taking the convex hull of $\{P_0, P_1, P_2\}$ get:



$$v(\alpha) < v(\beta)$$

ultrametric fact:

Case (A) $v(\alpha + \beta) = \min \{v(\alpha), v(\beta)\}$ if $v(\alpha) \neq v(\beta)$

Case (B) $v(\alpha + \beta) \geq v(\alpha) = v(\beta)$ if $v(\alpha) = v(\beta)$

$$y = mx + c$$

Case (A): $v(\alpha) \neq v(\beta)$

WLOG write $a = v(\alpha) \leq b = v(\beta)$. Then

$$P_0 = (0, a+b), P_1 = (1, a), P_2 = (2, 0)$$

We look at the line L passing through P_0 and P_2 .

its slope is $-\frac{a+b}{2} =: m$

using P_0 the equation should be

$$y = (a+b) + m(x-0)$$

$$= a+b - \frac{a+b}{2} x$$

The theory of the Newton Polygons can be done for power series.

Def 18 : Let $f \in \mathbb{Q}_K[[T]]$. Then $\text{New}_{\text{poly}}(f)$ is defined as the largest decreasing convex function below $\{(i, v(a_i))\}_{i \in \mathbb{Z}}$ i.e.

$$\mathcal{L}(\text{New}_{\text{poly}}(f))(r) = \begin{cases} v_r(f) := \inf \{v(a_i) + r \cdot i\}, & r \geq 0 \\ -\infty & , r < 0. \end{cases}$$

Remark :

A power series $f = \sum_{i \geq 0} a_i T^i$ with $a_i \in \mathbb{Q}_K$ converges on

$$\mathbb{D}_K^\circ = \{x \in \overline{K} \mid |x| < 1\} \iff v(x) > 0.$$

So we care about zeros with positive valuation.

Those zeros produce negative slopes in the Newton polygon (remember slope = $-v(\text{zero})$)

Zeros with $|x| > 1$, $v(x) < 0$, are outside $\mathbb{D}_K$
(Their slopes would be positive and we simply don't track them in this dis-geometry)

This is why we now demand the polygon be decreasing (all slopes ≤ 0) we are encoding only $v(x) \geq 0$ guys.

(iii) by Lemma 13, functions v_i are valuations.

Again the slopes of the Newton Polygon $\text{New}_{p,q}(f)$ of $f \in \mathbb{Q}_K[[T]]$ captures the valuation of the zeros of f .

Thm 19 (Lazard) :

Let $f \in \mathbb{Q}_K[[T]]$ and $\lambda \neq 0$ a slope of $\text{New}(f)$.
Then, there exists some $\alpha \in \overline{K}$ with $f(\alpha) = 0$
and $v(\alpha) = -\lambda$.