Local systems, Monodromy Representations and Connections.

X be a connected manifold with base point $x_0 \in X$.

Goal

$$\begin{tikzcd}
 & \mathrm{LocSys}_X^{\mathbb{C}} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathrm{Rep}_{\mathbb{C}}(\pi_1(X,x_0)) & & \mathrm{Conn}_X^{\nabla}
\end{tikzcd}$$

§1. Local systems and Monodromy Rep.

Cat 1: $\mathrm{LocSys}_X^{\mathbb{C}}$ Local systems.

Ob: Sheaves $\mathcal{F}$ of finite dimensional $\mathbb{C}$-v.sp. on X which are locally constant

[ie $\exists$ open cover $\{U_i\}$ of X s.t. $\mathcal{F}|_{U_i} \cong \mathbb{C}^r|_{U_i}$]

Mor: Morph. of sheaves of $\mathbb{C}$-v.spaces.

Cats 2: $\mathrm{Rep}_{\mathbb{C}}(\pi_1(X,x_0))$ Resps. of the fund. group.

Ob: Gp. homo. $\rho : \pi_1(X,x_0) \longrightarrow GL(V)$

where V is a fin. dim $\mathbb{C}$-v.sp.

$\pi_1(X,x_0) = \{$ loops based at x_0 up to homotopy $\}$

$\gamma(0) = \gamma(1) = x_0$

Mor: $\pi_1(X,x_0)$ - equivariant linear maps

[i.e. given $\rho_1 : \pi_1(X,x_0) \longrightarrow GL(V_1)$
$\rho_2 : \pi_1(X,x_0) \longrightarrow GL(V_2)$ Two reps of fun. gp.

A morphis between ρ_1 and ρ_2 is a $\mathbb{C}$-linear map $f : V_1 \longrightarrow V_2$ s.t

$$f(\rho_1(\gamma)(v)) = \rho_2(\gamma) f(v) \quad \forall \ \gamma \in \pi_1(X,x_0), \ v \in V_1$$

$$\begin{tikzcd}
V_1 \arrow[r, "\rho_1(\gamma)"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & V_1 \arrow[d, "f"] \\
V_2 \arrow[r, "\rho_2(\gamma)"'] & V_2
\end{tikzcd}$$

Theorem [Sab07, Thm 15.8]

Construction : $\mathcal{F} \longmapsto T^{\mathcal{F}}$

$\mathcal{F} \in \mathrm{LocSys}_X^{\mathbb{C}}$

$\gamma : [0,1] \longrightarrow X$ with base point x_0

We can find

(1) $\gamma([0,1]) \subset \bigcup_i \mathcal{U}_i \quad , \quad \mathcal{U}_i \underset{\text{open}}{\subset} X$

(2) Choose

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \sigma_{n+1} = 1$$

st

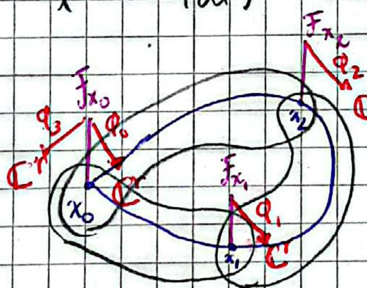
$$\mathcal{F}|\mathcal{U}_i \cong \underline{\mathbb{C}}^r|\mathcal{U}_i$$

$$\gamma([\sigma_i, \sigma_{i+1}]) \subset \mathcal{U}_i.$$

Set

$$\gamma_i : \gamma|_{[\sigma_i, \sigma_{i+1}]} : [\sigma_i, \sigma_{i+1}] \longrightarrow \mathcal{U}_i \quad , \quad \gamma(\sigma_i) = x_i$$

$\gamma_i^{-1}(\mathcal{F}|\mathcal{U}_i)$ is ~~locally constant sheaf~~ constant



Choose $\varphi_0 : \mathcal{F}_{x_0 = \gamma(\sigma_0)} = \left(\gamma^{-1}(\mathcal{F}|\mathcal{U}_0)\right)_{x_0} \longrightarrow \mathbb{C}^r$

$\Rightarrow \exists!\ \psi_0 : \gamma^{-1}(\mathcal{F}|\mathcal{U}_0) \longrightarrow \underline{\mathbb{C}}^r_{[\sigma_0, \sigma_1]}$

s.t. $(\psi_0)_{x_0} = \varphi_0$.

Define $\varphi_1 := (\psi_0)_{x_1} : \mathcal{F}_{x_1} \longrightarrow \mathbb{C}^r$

$\Rightarrow \exists!\ \psi_1 : \gamma^{-1}(\mathcal{F}|\mathcal{U}_1) \longrightarrow \underline{\mathbb{C}}^r_{[\sigma_1, \sigma_2]}$ s.t $(\psi_1)_{x_1} = \varphi_1$

Define

$\varphi_2 := (\psi_1)_{x_2} : \mathcal{F}_{x_2} \longrightarrow \mathbb{C}^r$

$\vdots$

$$\varphi_n := (\psi_{n-1})_{x_n} : \mathcal{F}_{x_n} \longrightarrow \mathbb{C}^r$$

$$\rightarrow \exists!\ \psi_n : \gamma^{-1}(\mathcal{F}|U_n) \rightarrow \underline{\mathbb{C}}^r_{[\sigma_n, \sigma_{n+1}]} \quad \text{s.t } (\psi_n)_n = \varphi_n$$

Define

$$\varphi_{n+1} : \mathcal{F}_{x_{n+1} = x_0} \longrightarrow \mathbb{C}^r$$

$$\left(\gamma(0) = \gamma(1) = 1 \right)$$

Thus

$$T_\gamma^{\mathcal{F}} : \varphi_{n+1} \circ \varphi_0^{-1} : \mathbb{C}^r \xrightarrow{\sim} \mathbb{C}^r$$

"parallel transport $v \in \mathbb{C}^r$ along γ".

$$T^{\mathcal{F}} : \begin{array}{ccc} \pi_1(X, x_0) & \longrightarrow & GL_r(\mathbb{C}) \\ \gamma & \longmapsto & T_\gamma^{\mathcal{F}} \end{array}$$

"Monodromy Reps"

Functoriality on the morph.

Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G} \in \mathrm{LocSys}_X^{\mathbb{C}}$

then for each $\mathcal{U} \underset{\text{open}}{\subseteq} X$, have $\mathbb{C}$-linear map

$$\varphi(\mathcal{U}) : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}) \quad \text{compa. with rest. maps}$$

i.e. $\forall\, V \subseteq \mathcal{U}$ the diagram

$$\begin{tikzcd} \mathcal{F}(\mathcal{U}) \arrow[r, "\varphi(\mathcal{U})"] \arrow[d, "res^{\mathcal{F}}_{V,\mathcal{U}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(\mathcal{U}) \arrow[d, "res^{\mathcal{G}}_{V,\mathcal{U}}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

commutes

which induces $\varphi_{x_0} : \mathcal{F}_{x_0} \longrightarrow \mathcal{G}_{x_0}$ linear map, $x_0 \in X$ base point.

Thus, $\forall\, \gamma \in \pi_1(X, x_0)$ the foll commutes

$$\begin{tikzcd} \mathcal{F}_{x_0} \arrow[r, "T^{\mathcal{F}}(\gamma)"] \arrow[d, "\varphi_{x_0}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}_{x_0} \arrow[d, "\varphi_{x_0}"] \\ \mathcal{G}_{x_0} \arrow[r, "T^{\mathcal{G}}(\gamma)"'] & \mathcal{G}_{x_0} \end{tikzcd}$$

$\Longrightarrow \varphi_{x_0}$ is a morph of mono reps

i.e. $\varphi_{x_0} \in \mathrm{Hom}_{\pi_1(X,x_0)}(T^{\mathcal{F}}, T^{\mathcal{G}})$

$$(\varphi : \mathcal{F} \longrightarrow \mathcal{G}) \longmapsto (\varphi_{x_0} : T^{\mathcal{F}} \longrightarrow T^{\mathcal{G}}).$$

Thus, $T^{(\cdot)} : \mathrm{LocSys}_X^{\mathbb{C}} \longrightarrow \mathrm{Rep}_{\mathbb{C}}(\pi_1(X, x_0))$

$$\mathcal{F} \longmapsto T^{\mathcal{F}}$$

$$\varphi \longmapsto \varphi_{x_0}$$

Construction $\rho \mapsto \mathcal{F}_\rho$

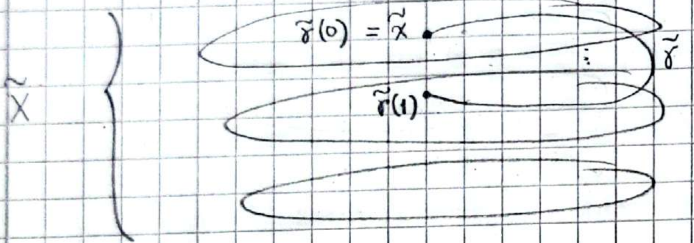
Set $\rho : \pi_1(X, x_0) \longrightarrow GL_r(\mathbb{C})$ be a represn of the fund gp

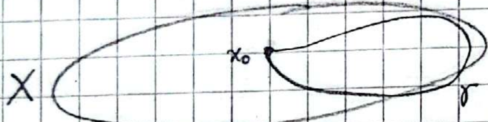
$$\left[\begin{array}{l} \rho(\gamma) : \mathbb{C}^r \longrightarrow \mathbb{C}^r \\ \qquad v \longmapsto \rho(\gamma)(v) \end{array}\right.$$, for some $\gamma \in \pi_1(X, x_0)$ with base point x_0

Consider $\pi : \tilde{X} \longrightarrow X$ ($\tilde{X}$ is simply connected).

Step 1 Action over $\tilde{X}$.

We choose $\tilde{x} \in \tilde{X}$ st $\pi(\tilde{x}) = x_0$. Then $\exists!$ $\tilde{\gamma}$ lift of γ st $\tilde{\gamma}(0) = \tilde{x}$.





$$\begin{tikzcd} \tilde{X} \arrow[rr, "{[\gamma]}"] \arrow[dr, "\pi"'] & & \tilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd} \qquad \tilde{x} \longmapsto \tilde{\gamma}(1)$$

We define

$$\pi_1(X, x_0) \times \tilde{X} \longrightarrow \tilde{X}$$

$$([\gamma], \tilde{x}) \longmapsto [\gamma] \cdot \tilde{x} := \tilde{\gamma}(1)$$ "Deck transformation"

Step 2 Action over $\tilde{X} \times \mathbb{C}^r$

Define a diagonal action

$$\pi_1(X, x_0) \times (\tilde{X} \times \mathbb{C}^r) \longrightarrow (\tilde{X} \times \mathbb{C}^r)$$

$$(\gamma, (\tilde{x}, v)) \longmapsto \gamma \cdot (\tilde{x}, v) = (\gamma \cdot \tilde{x}, \rho(\gamma) v).$$

Thus we get $E := (\tilde{X} \times \mathbb{C}^r) / \pi_1(X, x_0)$, $\quad \tilde{X} \times \mathbb{C}^r \xrightarrow{q} E$

"Set of orbits under the action"

Topology: take discrete top in $\mathbb{C}^r$ + define a top in $\tilde{X}$ $\Rightarrow$ Product topo + take the topology that makes q cont.

Notice that

$$E \xrightarrow{p} X$$ is well defi

$$\overline{(\tilde{x}, v)} \longmapsto p(\overline{(\tilde{x}, v)}) = \pi(\tilde{x})$$ because is the nat proj induced by π.

Step 3 $p : E \longrightarrow X$ is a vector bundle

$$\begin{tikzcd} \tilde{X} \times \mathbb{C}^r \arrow[r, "q"] \arrow[d, "proj"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\ \tilde{X} \arrow[r, "\pi"'] & X \end{tikzcd}$$

(1) $\forall\, x_0 \in X$, $E_{x_0} = p^{-1}(x_0)$ is a $\mathbb{C}$-v.sp. (Want $p^{-1}(x_0) \cong \mathbb{C}^r$)

$$p^{-1}(x_0) = \{ \overline{(\tilde{x}, v)} \in E \mid p(\overline{(\tilde{x}, v)}) = x_0 \}$$
$$= \{ \overline{(\tilde{x}, v)} \in E \mid \pi(\tilde{x}) = x_0 \}$$
$$= \{ \overline{(\tilde{x}, v)} \mid \tilde{x} \in \tilde{X},\ \pi(\tilde{x}) = x_0,\ v \in \mathbb{C}^r \}$$

// Recall / Fact: Deck transf. is transitive on the fibers:

$\forall\, \tilde{x}, \tilde{x}' \in \pi^{-1}(x_0)$, $\exists\, \gamma \in \pi_1(X, x_0)$ s.t. $\tilde{x}' = \gamma \cdot \tilde{x}$. //

So, we can fix $\tilde{y} \in \pi^{-1}(x_0)$. Then

$$p^{-1}(x_0) = \{ \overline{(\tilde{y}, v)} \mid v \in \mathbb{C}^r \}.$$

Indeed $\forall\, \tilde{x} \in \pi^{-1}(x_0)$, $\exists\, \gamma \in \pi_1(X, x_0)$ s.t. $\tilde{x} = \gamma \tilde{y}$.

Thus $\forall\, v \in \mathbb{C}^r$, $(\tilde{x}, v) = (\gamma \tilde{y}, v) = \gamma \cdot (\tilde{y}, \rho(\gamma^{-1}) v)$

$$= (\gamma \cdot \tilde{y}, \rho(\gamma)\rho(\gamma^{-1}) v)$$

So $\overline{(\tilde{x}, v)} = \overline{(\tilde{y}, \rho(\gamma) v)}$

- $\mathbb{C}^r \longrightarrow p^{-1}(x_0)$ is a bijection.
  $v \longmapsto \overline{(\tilde{y}, v)}$

Surjecti ✓

For the injectivity suppose $\overline{(\tilde{y}, v)} = \overline{(\tilde{y}, w)}$

Then, $\exists\, \gamma \in \pi_1(X, x_0)$, $(\tilde{y}, v) = \gamma \cdot (\tilde{y}, w) = (\gamma \tilde{y}, \rho(\gamma) w)$

In particular $\tilde{y} = \gamma \tilde{y}$ thus $\gamma = 1$ (see wikipedia).

2) $\exists$ open covering $\{\mathcal{U}_\alpha\}$ of X and homeomorphism

$$\begin{tikzcd} \varphi_\alpha : p^{-1}(\mathcal{U}_\alpha) \arrow[r] \arrow[dr] & \mathcal{U}_\alpha \times \mathbb{C}^r \arrow[d, "\text{proj}"] \\ & \mathcal{U}_\alpha \end{tikzcd} \quad \text{over } \mathcal{U}_\alpha$$

$(x,v) \mapsto x$

$\mathcal{U}_\alpha \subset X$ open

We take $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ to be the set of opens given by the property of the universal cover:

Recall $\forall x \in X$, $\exists\, \mathcal{U} \subset X$ with $x \in \mathcal{U}$ s.t.

$\pi^{-1}(\mathcal{U}) = \bigsqcup_i \mathcal{U}_i$ and $\pi|_{\mathcal{U}_i} : \mathcal{U}_i \longrightarrow \mathcal{U}$ is a homeomorphism.

Let $\mathcal{U}$ be such an open. We will show $p^{-1}(\mathcal{U}) \cong \mathcal{U} \times \mathbb{C}^r$

First $\quad p^{-1}(\mathcal{U}) \cong (\pi^{-1}(\mathcal{U}) \times \mathbb{C}^r) / \pi_1(X, x_0)$

$$\begin{aligned} p^{-1}(\mathcal{U}) &= \{ \overline{(\tilde{x},v)} \in E \mid p(\overline{(\tilde{x},v)}) \in \mathcal{U} \} \\ &= \{ \overline{(\tilde{x},v)} \in E \mid \pi(\tilde{x}) \in \mathcal{U} \} \\ &= \{ \overline{(\tilde{x},v)} \mid \tilde{x} \in \tilde{X},\ \pi(\tilde{x}) \in \mathcal{U},\ v \in \mathbb{C}^r \} \\ &= \{ \overline{(\tilde{x},v)} \mid \tilde{x} \in \pi^{-1}(\mathcal{U}),\ v \in \mathbb{C}^r \} \end{aligned}$$

$$\begin{aligned} (\pi^{-1}(\mathcal{U}) \times \mathbb{C}^r)/\pi_1(X,x_0) &= \{ \overline{(\tilde{x},v)} \mid \tilde{x} \in \pi^{-1}(\mathcal{U}),\ v \in \mathbb{C}^r \} \\ &= \{ \overline{(\tilde{x},v)} \mid \pi(\tilde{x}) \in \mathcal{U},\ v \in \mathbb{C}^r \} \end{aligned}$$

Second :

$$(\pi^{-1}(\mathcal{U}) \times \mathbb{C}^r)/\pi_1(X,x_0) = (\bigsqcup_i \mathcal{U}_i \times \mathbb{C}^r)/\pi_1(X,x_0)$$

Third :

$$(\bigsqcup_i \mathcal{U}_i \times \mathbb{C}^r)/\pi_1(X,x_0) \cong \mathcal{U} \times \mathbb{C}^r$$

$$(\bigsqcup_i \mathcal{U}_i \times \mathbb{C}^r)/\pi_1(X,x_0) = \{ \overline{(\tilde{x},v)} \in E \mid \tilde{x} \in \bigsqcup_i \mathcal{U}_i,\ v \in \mathbb{C}^r \}$$

We can fix $\mathcal{U}_i \subset \bigsqcup_i \mathcal{U}_i$ s.t. $(*) = \{ \overline{(\tilde{x},v)} \in E \mid \tilde{x} \in \mathcal{U}_i,\ v \in \mathbb{C}^r \}$

for $(*)$ Indeed, $\tilde{x} \in \bigsqcup_i \mathcal{U}_i$, $\exists\, \gamma \in \pi_1(X,x_0)$, s.t. $\gamma\cdot\tilde{x} \in \mathcal{U}_i$ (Transitivity of Deck transf.)

Thus, $\forall v \in \mathbb{C}^r$

$$\begin{aligned} (\tilde{x},v) &= \gamma^{-1}(\gamma\cdot\tilde{x},\ \rho(\gamma)v) \\ &= (\gamma^{-1}\gamma\tilde{x},\ \rho(\gamma^{-1})\rho(\gamma)v) \end{aligned}$$

$$\Rightarrow \overline{(\tilde{x},v)} = \overline{(\gamma\cdot\tilde{x},\ \rho(\gamma)v)}$$

Want:

$$\mathcal{U}_\lambda \times \mathbb{C}^r \longrightarrow \left(\bigsqcup \mathcal{U}_i \times \mathbb{C}^r\right)/\pi_1(X,x_0) \qquad \text{bijection}$$

$$(x,v) \longmapsto \overline{(\pi^{-1}|_{\mathcal{U}_\lambda}(x)\,,\, v)} \qquad (=\overline{(\tilde{x},v)})$$

Surjectivity ✓

Injectivity: Suppose $\overline{(\pi^{-1}|_{\mathcal{U}_\lambda}(x), v)} = \overline{(\pi^{-1}|_{\mathcal{U}_\lambda}(x'), v')}$.

Then $\exists\, \gamma \in \pi_1(X,x_0)$ s.t.

$$(\pi^{-1}|_{\mathcal{U}_\lambda}(x), v) = \gamma\cdot(\pi^{-1}|_{\mathcal{U}_\lambda}(x'), v)$$

$$= (\gamma\cdot\pi^{-1}|_{\mathcal{U}_\lambda}(x'), \rho(\gamma)v)$$

$$\Rightarrow \pi^{-1}|_{\mathcal{U}_\lambda}(x) = \gamma\cdot\pi^{-1}|_{\mathcal{U}_\lambda}(x')$$

$$\Rightarrow x = \pi(\pi^{-1}|_{\mathcal{U}_\lambda}(x)) = \pi(\gamma\cdot\pi^{-1}|_{\mathcal{U}_\lambda}(x')) = x'$$

and $v = \rho(\gamma)v'$, because $\pi^{-1}|_{\mathcal{U}_\lambda}(x) = \gamma\,\pi^{-1}|_{\mathcal{U}_\lambda}(x)$

$\Rightarrow \underset{\text{(wikipedia)}}{\gamma = 1}$ and $v = v'$. ∎

Thus

$$p^{-1}(\mathcal{U}) \cong \left(\pi^{-1}(\mathcal{U})\times\mathbb{C}^r\right)/\pi_1(X,x_0) \cong \left(\bigsqcup \mathcal{U}_i\times\mathbb{C}^r\right)/\pi_1(X,x_0) \cong \mathcal{U}\times\mathbb{C}^r.$$

Then, we can define a section of the vector bundle $E \xrightarrow{p} X$ over $\mathcal{U} \underset{\text{open}}{\subset} X$ as a continuous map $s:\mathcal{U}\to E$ s.t. $p\circ s = id_{\mathcal{U}}$.

Define the sheaf of sections of $E\xrightarrow{p} X$ by

$$\mathcal{F}(\mathcal{U}) := \{\, s:\mathcal{U}\to E \mid p\circ s = id_{\mathcal{U}} \,\}.$$

Step 4: $\mathcal{F} \in LocSys^{\mathbb{C}}_X$ (i.e. $\exists \{\mathcal{U}_\alpha\}$ of X s.t. $\mathcal{F}|_{\mathcal{U}} \cong \mathbb{C}^r$)

Take $\mathcal{U}$ as before and connected, then we have a trivialization

$$p^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}\times\mathbb{C}^r$$

$$\begin{tikzcd} E \arrow[r, "\sim"] \arrow[dr, "p"'] & \mathcal{U}\times\mathbb{C}^r \arrow[d, "proj"] \\ & \mathcal{U} \end{tikzcd}$$

Have

$$\mathcal{F}(\mathcal{U}) := \{\, s: \mathcal{U} \longrightarrow \mathcal{U}\times\mathbb{C}^r \mid proj\circ s = id_{\mathcal{U}} \,\}$$

$$x \longmapsto s(x) = (a,b), \quad proj(s(x)) = x$$

Since $proj(a,b) = a$ then

$$:= \{\, s:\mathcal{U}\to\mathbb{C}^r \mid s \text{ cont} \,\}.$$

Since $\mathcal{U}$ connect, $\mathbb{C}^r$ discr. top, s cont $\Rightarrow s$ is constant.

$\Rightarrow \mathcal{F}(\mathcal{U}) = \mathbb{C}^r \Rightarrow \mathcal{F}|_{\mathcal{U}} \cong \mathbb{C}^r$.

Functionality on the morphisms:

We want

$$\mathrm{Rep}_{\mathbb{C}}(\pi_1(X, x_0)) \longrightarrow \mathrm{Loc\,Sys}^{\mathbb{C}}_X$$
$$(f : (\mathbb{C}^r, \rho_1) \to (\mathbb{C}^s, \rho_2)) \longmapsto (\varphi : \mathcal{F}_{\rho_1} \to \mathcal{F}_{\rho_2}).$$

Consider

$$\rho_1 : \pi_1(X, x_0) \longrightarrow GL_r(\mathbb{C})$$
$$\gamma \longmapsto \rho_1(\gamma) : \mathbb{C}^r \to \mathbb{C}^r \qquad \text{two reps}$$
$$\rho_2 : \pi_1(X, x_0) \longrightarrow GL_s(\mathbb{C})$$
$$\gamma \longmapsto \rho_2(\gamma) : \mathbb{C}^s \to \mathbb{C}^s$$
$$v \longmapsto \rho_2(\gamma)(v)$$

and a morphism $f : \mathbb{C}^r \longrightarrow \mathbb{C}^s$ i.e
$$v \longmapsto f(v)$$

$$\begin{array}{ccc} \mathbb{C}^r & \xrightarrow{\rho_1(\gamma)} & \mathbb{C}^r \\ f\downarrow & \circlearrowleft & \downarrow f \\ \mathbb{C}^s & \xrightarrow[\rho_2(\gamma)]{} & \mathbb{C}^s \end{array} \qquad \begin{array}{l} \rho_2(\gamma) \circ f = f \circ \rho_1(\gamma), \quad \forall \gamma \in \pi_1(X, x_0). \\ \rho_2(\gamma)(f(v)) = f(\rho_1(\gamma)(v)) \end{array}$$

Define the map:

$$\tilde{f} : \tilde{X} \times \mathbb{C}^r \longrightarrow \tilde{X} \times \mathbb{C}^s$$
$$(\tilde{x}, v) \longrightarrow (\tilde{x}, f(v))$$

is well defined because f is morphis of reps

Note that we have

$\tilde{X} \times \mathbb{C}^r$ is a $\pi_1(X, x_0)$-Set i.e. $\pi_1(X, x_0)$ acts on $\tilde{X} \times \mathbb{C}^r$

$\tilde{X} \times \mathbb{C}^s$ is a $\pi_1(X, x_0)$-Set $\pi_1(X, x_0)$ acts on $\tilde{X} \times \mathbb{C}^s$

Recall

$$\pi_1(X, x_0) \times (\tilde{X} \times \mathbb{C}^r) \longrightarrow (\tilde{X} \times \mathbb{C}^r)$$
$$(\gamma, (\tilde{x}, v)) \longmapsto \gamma \cdot (\tilde{x}, v) = (\gamma \cdot \tilde{x}, \rho_1(\gamma) v),$$
$$\pi_1(X, x_0) \times (\tilde{X} \times \mathbb{C}^s) \longrightarrow (\tilde{X}, \mathbb{C}^s)$$
$$(\gamma, (\tilde{x}, f(v))) \longmapsto \gamma \cdot (\tilde{x}, f(v)) = (\gamma \cdot \tilde{x}, \rho_2(\gamma) f(v)).$$

Wan to check if $\tilde{f} : \tilde{X} \times \mathbb{C}^r \longrightarrow \tilde{X} \times \mathbb{C}^s$ is a function of $\pi_1(X, x_0)$-sets

then $\tilde{f}(\gamma \cdot (\tilde{x}, v)) = \gamma \cdot \tilde{f}((\tilde{x}, v)) = \gamma \cdot (\tilde{x}, f(v))$.

Indeed:

$$\begin{aligned} \tilde{f}(\gamma \cdot (\tilde{x}, v)) &= \tilde{f}(\gamma \cdot \tilde{x}, \rho_1(\gamma) v) \\ &= (\gamma \cdot \tilde{x}, f(\rho_1(\gamma) v)) \\ &= (\gamma \cdot \tilde{x}, \rho_2(\gamma)(f(v))) \\ &= \gamma \cdot (\tilde{x}, f(v)). \end{aligned}$$

then, it induces, a map on the level of quotients

$$F(f): E_{\rho_1} = (\tilde{X} \times \mathbb{C}^r) / \pi_1(X, x_0) \longrightarrow E_{\rho_2} = (\tilde{X} \times \mathbb{C}^s) / \pi_1(X, x_0)$$

$$\overline{(\tilde{x}, v)} \longmapsto \overline{(\tilde{x}, f(v))} = F(f)(\overline{(\tilde{x}, v)})$$

and we have the identification

$$\gamma \cdot (\tilde{x}, v) = (\gamma \tilde{x}, \rho_1(\gamma) v) \longleftrightarrow \gamma \cdot (\tilde{x}, f(v)) = (\gamma \tilde{x}, \rho_2(\gamma) f(v)).$$

! well defi.

We get a commutative diagram

$$\begin{tikzcd}
E_{\rho_1} \arrow[rr, "F"] \arrow[dr, "p_1"'] & & E_{\rho_2} \arrow[dl, "p_2"] \\
& X \arrow[ul, bend left, dashed, "s"] \arrow[ur, bend right, dashed, "s'"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$$p_2(F(\overline{(\tilde{x}, v)})) = p_2(\overline{(\tilde{x}, f(v))}) = \pi(\tilde{x})$$

$$p_1(\overline{(\tilde{x}, v)}) = \pi(\tilde{x}) \qquad \lrcorner$$

This induces a morphism of sheaves of $\mathbb{C}$-v.sp defined by postcomposition

$$\mathcal{F}(f): \mathcal{F}_{\rho_1} \longrightarrow \mathcal{F}_{\rho_2}, \quad \mathcal{F}(f)(s) := F(f) \circ s$$

i.e. $\mathcal{F}(f)(s): \mathcal{U} \longrightarrow E_{\rho_2}$

where

$$\mathcal{F}_{\rho_1}(\mathcal{U}) := \{ s: \mathcal{U} \longrightarrow E_{\rho_1} \mid p_1 \circ s = id_{\mathcal{U}} \}$$

$$\mathcal{F}_{\rho_2}(\mathcal{U}) := \{ s': \mathcal{U} \longrightarrow E_{\rho_2} \mid p_2 \circ s' = id_{\mathcal{U}} \}.$$

We have

$$\begin{aligned}
p_2(F(f)(s(x))) &= p_2(F(f)(\overline{(\tilde{x}, v)})) \\
&= p_2(\overline{(\tilde{x}, f(v))}) \\
&= \pi(\tilde{x}) = x
\end{aligned}$$

This means $p_2 \circ F(f) \circ s = p_1 \circ s = id_{\mathcal{U}}$.

"Because restriction commutes with postcompo."

Thus $F(f) \circ s$ is a section in $\mathcal{F}_{\rho_2}(\mathcal{U})$.

Then we got a well defined morphism of sheaves $\mathcal{F}(f): \mathcal{F}_{\rho_1} \longrightarrow \mathcal{F}_{\rho_2}$.

Note that we can define $\mathcal{F}(id) = id$ and $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$,

which follows from

$$\begin{aligned}
\mathcal{F}(f \circ g)(s) = F(f \circ g)(s(x)) = F(f \circ g)(\overline{(\tilde{x}, v)}) &= \overline{(\tilde{x}, (f \circ g)(v))} \\
&= \overline{(\tilde{x}, f(g(v)))} \\
&= F(f)\, \overline{(\tilde{x}, g(v))} = \mathcal{F}(f)\, \mathcal{F}(g)(s(x)) \\
&= \mathcal{F}(f) \circ \mathcal{F}(g)(s)
\end{aligned}$$

Thus, $f \longmapsto F(f)$ and we get a functor

$$\begin{aligned} F_{(\,)} : \mathrm{Rep}_{\mathbb{C}}(\pi_1(X,x)) &\longrightarrow \mathrm{LocSys}_X^{\mathbb{C}} \\ \rho &\longmapsto F_\rho \\ f &\longmapsto F(f) \end{aligned}$$

§ 2. Local systems and flat connection.

Def: $E \to X$ holomorphic vector bundle over X cx manifold.
$\mathcal{E}$ its associated sheaf of sections.

(1) A holomorphic connection ∇ on $E \to X$ is a morph of sheaves

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \qquad \mathbb{C}\text{-linear}$$

Satisfying Leibniz rule: for any $\mathcal{U} \underset{\text{open}}{\subset} X$, any $s \in \mathcal{E}(\mathcal{U})$,

$$\nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s \in \Omega^1_X(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{E}(\mathcal{U}).$$

"We say (E, ∇) is a holomorphic vector bundle with conn. ∇ on X"

(2) ∇ is said to be integrable or flat if $\nabla^2 = 0$. $\mathcal{U} \underset{\text{open}}{\subset} X$

$$\mathcal{E}(\mathcal{U}) \xrightarrow{\nabla} \Omega^1_X(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{E}(\mathcal{U}) \xrightarrow{\nabla} \Omega^2_X(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{E}(\mathcal{U})$$

$$s \longmapsto \nabla(s) := \omega \otimes s \longmapsto \nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s)$$

$$= \underset{\Omega^2_X(\mathcal{U})}{d\omega} \otimes \underset{\mathcal{E}(\mathcal{U})}{s} - \underset{\Omega^2_X(\mathcal{U})}{\omega \wedge \omega} \otimes \underset{\mathcal{E}(\mathcal{U})}{s}.$$

$d : \Omega^1_X(\mathcal{U}) \to \Omega^2_X(\mathcal{U})$, $\omega \mapsto d\omega$. via

Cat 3 Conn^{∇}_X

Obj: (E, ∇)

Mor: a morph between (E, ∇) and (E, ∇') is a morph $E \to E'$ which induces a morph. $\mathcal{E} \xrightarrow{\varphi} \mathcal{E}'$ $\mathcal{O}_X$-linear

s.t.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \\ {\scriptstyle\varphi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle id \otimes \varphi} \\ \mathcal{E}' & \xrightarrow[\nabla']{} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}' \end{array}$$

$\nabla' \circ \varphi = (id \otimes \varphi) \circ \nabla$.

ie φ is compatible with connec

Theorem [Del 70, I, Thm 2.17]
There exists

$$\mathrm{LocSys}^{\mathbb{C}}_X \rightleftarrows \mathrm{Conn}^{\nabla}_X$$

"Proof" [Sab 07, Thm 12.8] (Cauchy–Kowalevski Theorem)

I Construction $\mathcal{F} \longmapsto (E, \nabla)$

Let $\mathcal{F} \in \mathrm{LocSys}_X^r$ (a local system of rank r over a complex manifold X).

Step 1 : Construct the vector bundle

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X.

Define
$$\mathcal{E} := \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}.$$

Since $\mathcal{F}$ is locally constant sheaf of rank r, we can write $\mathcal{F}|_{\mathcal{U}} \cong \mathbb{C}^r$

Then,
$$\mathcal{E}(\mathcal{U}) = \mathcal{O}_X(\mathcal{U}) \otimes_{\mathbb{C}} \mathbb{C}^r \cong \mathcal{O}_X(\mathcal{U})^r$$

thus, $\mathcal{E}$ is a locally free sheaf of rank r.

Step 2 : Define the connection

Let $\mathcal{E} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}$ (constructed from the local system $\mathcal{F}$).

Define $\nabla : \mathcal{E} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ on sections over any $\mathcal{U} \underset{\text{open}}{\subset} X$

by the rule $\nabla(f \otimes s) := df \otimes s$, $\forall f \in \mathcal{O}_X(\mathcal{U})$; $s \in \mathcal{F}(\mathcal{U})$.

This rule extends by $\mathbb{C}$-linearity to arbitrary section of $\mathcal{E}(\mathcal{U})$.

Verify Leibniz Rule : Let $h \in \mathcal{O}_X(\mathcal{U})$, then

$$\begin{aligned}
\nabla(h \cdot (f \otimes s)) &= \nabla(hf \otimes s) \\
&= d\,hf \otimes s \\
&= h \cdot df \otimes s + f \cdot dh \otimes s \\
&= h \cdot \nabla(f \cdot s) + dh \otimes (f \cdot s).
\end{aligned}$$

$$\mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{O}_X(\mathcal{U}) \otimes \mathcal{F}(\mathcal{U})$$
$$s \longmapsto (1 \otimes s)$$
$$\nabla(1 \otimes s) = d1 \otimes s = 0$$

∇ is flat :

$$\nabla(df \otimes s) = \underbrace{dd f}_{0} \otimes s - df \wedge \underbrace{\nabla(s)}_{0}$$

Want $\nabla^2 = 0$ Take $\mathcal{U} \underset{\text{open}}{\subseteq} X$, and $\mathcal{E}(\mathcal{U}) = \mathcal{O}_X(\mathcal{U}) \otimes_{\mathbb{C}} \mathcal{F}(\mathcal{U})$

$$\mathcal{O}_X(\mathcal{U}) \otimes \mathcal{F}(\mathcal{U}) \xrightarrow{\nabla} \Omega^1_X(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{E}(\mathcal{U}) \xrightarrow{\nabla} \Omega^2_X(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{E}(\mathcal{U})$$

$$f \otimes s \longmapsto \nabla(f \otimes s) = df \otimes s$$

$$\nabla(df \otimes s) = \overset{0}{\cancel{d\,df}} \otimes s - df \wedge \nabla(s)$$

$$\cancel{\neq ddf \otimes s - df \wedge f \otimes s}$$

(?)

$$\mathcal{F}(\mathcal{U}) \hookrightarrow \mathcal{O}_X(\mathcal{U}) \otimes \mathcal{F}(\mathcal{U})$$

$$s \longmapsto 1 \otimes s$$

$$\nabla(1 \otimes s) = d1 \otimes s = 0$$

$[df \wedge 0 = 0]$

[Local description II.1] So far we have shown an abstract description of ∇ for a general $\mathcal{U} \underset{\text{open}}{\subseteq} X$. Now we will take the particular $\mathcal{U}$ which trivializes $\mathcal{E}$ and give the local description of ∇.

Let $\mathcal{U} \underset{\text{open}}{\subseteq} X$ be an open such that $\mathcal{E}_{|\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}^r$

"[The connection that we have contructed is]" Now

$$\nabla_{|\mathcal{U}} : \mathcal{O}^r_{\mathcal{U}} \longrightarrow \Omega^1_{\mathcal{U}} \otimes_{\mathcal{O}_{\mathcal{U}}} \mathcal{O}^r_{\mathcal{U}} \cong (\Omega^1_{\mathcal{U}})^r$$

"$(j \longleftrightarrow i)$"

Take $\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \in \mathcal{O}^r_X(\mathcal{U})$ with $f_i \in \mathcal{O}_X(\mathcal{U})$.

We have $\vec{f} = \sum_{i=1}^{r} f_i e_i$ where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$.

Have $\nabla(e_i) = \begin{pmatrix} \omega_{1i} \\ \vdots \\ \omega_{ri} \end{pmatrix} \in (\Omega^1_X(\mathcal{U}))^r$ and define $A = (-1)(\omega_{ji})_{ji} \in M_{r\times r}(\Omega^1_X(\mathcal{U}))$

"connection matrix"

$$\nabla(\vec{f}) = \nabla\left(\sum_{i=0}^{r} f_i e_i\right) = \sum_{i=0}^{r} \nabla(f_i e_i) = \sum_{i=0}^{r} df_i \otimes e_i + f_i \cdot \nabla(e_i)$$

$$= \underbrace{\begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix}}_{= d\vec{f}} + (-1) A \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$$

[Local description $\pi-2-$]

for $\mathcal{U} \subset X$ open

Def $f \in \mathcal{E}(\mathcal{U})$ is ∇-horizontal if $\nabla(f) = 0$ in $\Omega^1_X(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{E}(\mathcal{U})$

Denote

$$\operatorname{Ker}\nabla(\mathcal{U}) = \{ f \in \mathcal{E}(\mathcal{U}) \mid \nabla(f) = 0 \}.$$

Coming back

For $\mathcal{U} \underset{\text{open}}{\subseteq} X$ s.t. $\mathcal{E}|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}^r$,

Have $\vec{f} \in \operatorname{Ker}\nabla(\mathcal{U}) \iff \begin{pmatrix} d f_1 \\ \vdots \\ d f_r \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$. (1)

Recall/Fact: In local coordinates $z_1, \dots, z_r$

- $\forall \omega \in \Omega^1_X(\mathcal{U})$, $\exists \varphi_i \in \mathcal{O}_X(\mathcal{U})$ s.t

$$\omega = \sum_i \varphi_i \, dz_i,$$

Thus $A = \sum_i A_i \cdot dz_i$ for some $A_i \in M_{r\times r}(\mathcal{O}_X(\mathcal{U}))$

- $\forall f \in \mathcal{O}_X(\mathcal{U})$, Have $df = \sum_i \left(\frac{\partial f}{\partial z_i}\right) dz_i$

We can rewrite (1) as follows.

$$\begin{pmatrix} d f_1 \\ \vdots \\ d f_r \end{pmatrix} = \sum_i \begin{pmatrix} \frac{\partial f_1}{\partial z_i} \\ \vdots \\ \frac{\partial f_r}{\partial z_i} \end{pmatrix} d z_i \iff \frac{\partial \vec{f}}{\partial z_i} = A_i \vec{f}, \ \forall i \in \{1, \dots, r\}$$

$$= \left(\sum_i A_i \cdot dz_i\right) \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$$

"This is a system of first order linear PDE's whose solution describe flat sections of the flat connection".

Comment See Kat

In case $A(z) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & \cdots & 0 & -1 \\ a_n(z) & & \cdots & & a_1(z) \end{pmatrix}$

The kernel corresponds to a solution of linear homog. equat.

$$\left(\frac{d}{dz}\right)^n f + a_1(z) \cdot \left(\frac{\partial}{\partial z}\right)^{n-1} f + \dots + a_n(z) f = 0.$$

III $(E, \nabla) \longmapsto E^{\nabla} := \operatorname{Ker} \nabla$
+ flat

$\operatorname{Ker} \nabla(\mathcal{U}) = \{ \vec{f} \in \mathcal{O}_X(\mathcal{U})^r \mid \frac{\partial \vec{f}}{\partial z_i} = A_i \vec{f}, \forall i = 1, \dots, n \}$

$\operatorname{Ker} \nabla$ is locally constant?

Take $\mathcal{U} \cong \mathbb{D}^n \subseteq \mathbb{C}^n$.

$\forall$ choices $v \in \mathbb{C}^r$, $\exists! \ \vec{f}_v \in \mathcal{O}_X(\mathcal{U})^r$ s.t. $\vec{f}_v \in \operatorname{Ker} \nabla(\mathcal{U})$

and $\vec{f}_v(0) = v$ (Cauchy condition).

Thus, we got

$$\begin{aligned} \mathbb{C}^r &\longrightarrow \operatorname{Ker} \nabla(\mathcal{U}) \\ v &\longmapsto \vec{f}_v \\ \vec{f}(0) &\longleftarrow \vec{f} \end{aligned}$$

Let $x_0 \in X$. Since X is a connected manifold and E is a holomorphic v. bundle, $\exists$ a small $\underset{x_0 \in}{\mathcal{U}}$ (e.g. $\mathcal{U} \cong \mathbb{D}^n$) s.t. $E|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}^r$, $r = \operatorname{rank}$ of E.

Moreover the connection ∇ is given by a connection matrix $A = \sum_{l=1}^{n} A^{(l)} dz_l$ where $A^{(l)} \in M_{r \times r}(\mathcal{O}_X(\mathcal{U}))$.

Given a local section $\vec{f} \in \mathcal{O}_X^r(\mathcal{U})$, the connection acts as

$$\nabla(\vec{f}) = d\vec{f} - A\vec{f}$$

Thus, $\vec{f}$ is horizontal ($\nabla(\vec{f}) = 0$) iff it satisfies the system of first order PDE's $d\vec{f} = A\vec{f}$

In local coordinates $z_1, \dots, z_n$: $\frac{\partial \vec{f}}{\partial z_k} = A^{(k)} \vec{f}$, $\forall k = 1, \dots, n$

(This is a system of r equation for r unknown hol function $f_1, \dots, f_r$)

Now, since ∇ is flat

- $\nabla^2 = 0$

$\Updownarrow$ equivalently: the matrixes $A^{(k)}$ satisfy the condition

$$\frac{\partial A^{(l)}}{\partial z_k} - \frac{\partial A^{(k)}}{\partial z_l} + [A^{(l)}, A^{(k)}] = 0, \ \forall k, l$$

Thus, from the system $\frac{\partial \vec{f}}{\partial z_k} = A^{(k)} \vec{f}$ we can have:

(*)

Exercise 12.6

Setup: If the connection is given locally by

$$\nabla(\vec{f}) = d\vec{f} - A\vec{f} \quad \text{with} \quad A = \sum_{l=1}^{n} A^{(l)} dz_l , \qquad \text{Then}$$

the flatness condition $dA + A \wedge A = 0$ can be written in coordinates

$$\frac{\partial A^{(l)}}{\partial z_k} - \frac{\partial A^{(k)}}{\partial z_l} = [A^{(l)}, A^{(k)}] , \quad \forall k, l$$

where

$$[A^{(l)}, A^{(k)}] = A^{(l)} A^{(k)} - A^{(k)} A^{(l)} \quad \text{(commutator)}$$

Sol:

Recall: · A is a matrix of holomorphic 1-forms

· $A^{(l)}$ is the coefficient matrix for dz_l

· dA means taking the differential of each entry of A

· $A \wedge A$ is matrix multipl + wedge products

So

* $$dA = \sum_{l=1}^{n} dA^{(l)} \wedge dz_l = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial A^{(l)}}{\partial z_k} dz_k \right) \wedge dz_l = \sum_{k,l} \frac{\partial A^{(l)}}{\partial z_k} dz_k \wedge dz_l$$

* $$A \wedge A = \left(\sum_{l=1}^{n} A^{(l)} dz_l \right) \wedge \left(\sum_{k=1}^{n} A^{(k)} dz_k \right)$$

$$= \sum_{l,k} A^{(l)} A^{(k)} dz_l \wedge dz_k$$

Recall:

$dz_l \wedge dz_k = -dz_k \wedge dz_l$, (antisim)

$dz_l \wedge dz_l = 0$

$$\Rightarrow dA + A \wedge A = \sum_{k,l} \Big(\underbrace{\frac{\partial A^{(l)}}{\partial z_k} dz_k \wedge dz_l}_{} + \underbrace{A^{(l)} A^{(k)} dz_l \wedge dz_k}_{- A^{(l)} A^{(k)} dz_k \wedge dz_l} \Big)$$

fixing (k,l)

Have

· $\dfrac{\partial A^{(l)}}{\partial z_k} dz_k \wedge dz_l$

· $-\dfrac{\partial A^{(k)}}{\partial z_l} dz_k \wedge dz_l$

$$\rightarrow \frac{\partial A^{(l)}}{\partial z_k} - \frac{\partial A^{(k)}}{\partial z_l} + A^{(l)} A^{(k)} - A^{(k)} A^{(l)} \quad dz_k \wedge dz_l$$

-1-

<u>Example</u> $X = \mathbb{C}^\times$

Take $z \frac{df}{dz} = \alpha f$, $\alpha \in \mathbb{C}$ ODE.

It defines a sheaf on X. $\mathcal{U} \underset{\text{open}}{\subset} X$

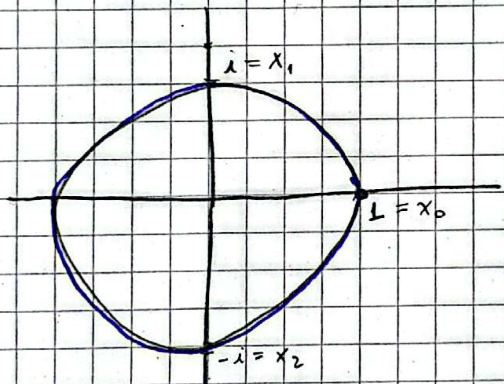
$$E^\nabla(\mathcal{U}) = \{ f : \mathcal{U} \to \mathbb{C} \mid z \frac{df}{dz} = \alpha f \}$$

locally constant of rank 1.

Want:

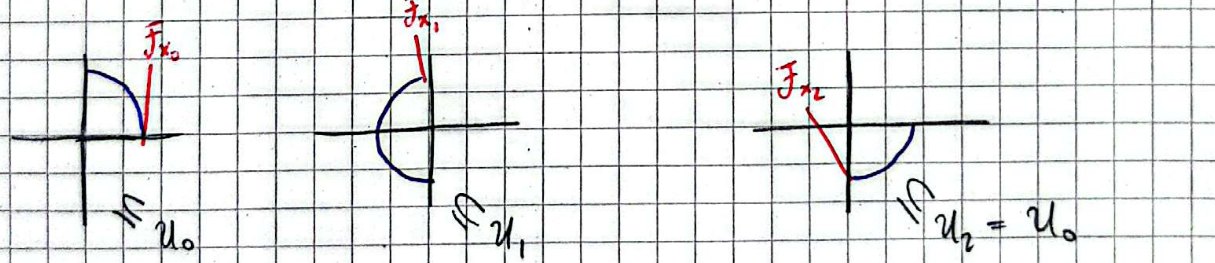
We will construct $T^{E^\nabla} \in Rep_{\mathbb{C}}(\pi_1(X, z_0))$

$$\pi_1(\mathbb{C}^\times, 1) = \mathbb{Z} \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$$



$\mathcal{U}_0 = \mathbb{C}^\times \setminus \mathbb{R}_{<0}$

$\mathcal{U}_1 = \mathbb{C}^\times \setminus \mathbb{R}_{>0}$



Have

$$\gamma : [0,1] \longrightarrow \mathbb{C}^\times$$
$$\sigma \longmapsto e^{i 2\pi \sigma}$$

$$0 = \sigma_0 < \sigma_1 = 1/4 < \sigma_2 = 3/4 < \sigma_3 = 1$$

Have

$$E^{\nabla}|_{U_0} \simeq \underline{\mathbb{C}}|_{U_0} \cdot e_0$$

where
$$e_0 : U_0 \longrightarrow \mathbb{C}$$
$$z \longmapsto \exp(\alpha \log_{U_0}(z))$$

with $\log_{U_0}(z) = \ln r + i\theta$

for $\theta \in (-\pi, \pi)$ and $z = re^{i\theta}$

Choose:

$$\varphi_0 : \left(E^{\nabla}|_{U_0}(U)\right)_{x_0} = \{\lambda e_0 \mid \lambda \in \mathbb{C}\} \xrightarrow{\sim} \mathbb{C}$$
$$e_0 \longmapsto e_0(1) = 1$$

$\exists!$ $\quad \Psi_0 : \gamma^{-1}(E^{\nabla}|_{U_0}) \longrightarrow \underline{\mathbb{C}}_{[\sigma_0, \sigma_1]}$

s.t. $\quad (\Psi_0)_{\sigma_0} = \varphi_0$

Define

$$\varphi_1 := (\Psi_0)_{\sigma_1} : E^{\nabla}_{x_1} \longrightarrow \mathbb{C}$$

$\exists!$ $\quad \Psi_1 : \gamma^{-1}(E^{\nabla}|_{U_1}) \longrightarrow \underline{\mathbb{C}}_{[\sigma_1, \sigma_2]}$

s.t. $\quad (\Psi_1)_{\sigma_1} = \varphi_1$

Let $\quad e_1 : U_1 \longrightarrow \mathbb{C}$
$$z \longmapsto \exp(\alpha \log_{U_1}(z))$$

$\log_{U_1}(z) = \ln r + i\theta$ with $\theta \in (0, 2\pi)$, $z = re^{i\theta}$

Have $E^{\nabla}|_{U_1} \simeq \underline{\mathbb{C}} \cdot e_1$, thus $\underset{(\text{"}\gamma(\sigma_1))}{E^{\nabla}_{x_1}} \simeq \mathbb{C} e_1$

Since
$$e_0|_{U_0 \cap U_1} = e_1|_{U_0 \cap U_1},$$

$$(\Psi_1)_{\sigma_1}(e_1) = (\Psi_1)_{\sigma_1}(e_0) = \varphi_1(e_0) = 1$$

We define

$$\varphi_2 := (\Psi_1)_{\sigma_2} : \underset{\substack{\| \\ \mathbb{C}\cdot e_1}}{E^{\nabla}_{x_2}} \longrightarrow \mathbb{C}$$
$$e_1 \longmapsto 1$$

$\exists!\ \Psi_2 : \gamma^{-1}(E^{\nabla}|_{\mathcal{U}_2}) \longrightarrow \underline{\mathbb{C}}_{[\sigma_2, 1]}$ s.t. $(\Psi_2)_{\sigma_2} = \varphi_2$

Set $e_2 : \mathcal{U}_2 \longrightarrow \mathbb{C}$
$$z \longmapsto \exp(\alpha \log_{\mathcal{U}_2}(z))$$

$$\log_{\mathcal{U}}(z) = \ln r + i\theta \quad , \theta \in (\pi, 3\pi) \quad , \quad z = re^{i\theta}$$

Have $E^{\nabla}|_{\mathcal{U}_2} \cong \underline{\mathbb{C}} \cdot e_2$, thus $E^{\nabla}_{x_0} \cong \mathbb{C} e_2$.

Since $e_1|_{\mathcal{U}_1 \cap \mathcal{U}_2} = e_2|_{\mathcal{U}_1 \cap \mathcal{U}_2}$,

$$(\Psi_2)_{\sigma_2}(e_2) = (\Psi_2)_{\sigma_2}(e_1) = \varphi_2(e_1) = 1$$

Define

$$\varphi_3 =: (\Psi_2)_{\sigma_3} : E^{\nabla}_{x_3} \longrightarrow \mathbb{C}$$
$$\left(\mathbb{C}\cdot e_2 \ \| \right) \ e_2 \longmapsto 1$$

Thus

$$T_{\gamma}^{E^{\nabla}} : \varphi_3 \circ \varphi_0^{-1} : \mathbb{C} \longrightarrow E^{\nabla}_1 \longrightarrow \mathbb{C}$$
$$1 \longmapsto \underset{\substack{\| \\ e^{-\alpha i 2\pi} e_2}}{e_0} \longmapsto e^{-\alpha i 2\pi}$$

$$\frac{e_0(1)}{e_2(1)} = \frac{1}{e^{\alpha i 2\pi}}$$