Analytic Rings

Introduction

Having replaced topological abelian groups by condensed abelian groups, we introduced *solid abelian groups* to obtain a notion of "complete" abelian groups. The corresponding category $\mathtt{Solid} \hookrightarrow \mathtt{Cond}(\mathtt{Ab})$ is (very) well-behaved:

- $\mathtt{Solid}$ is an abelian subcategory, stable under limits, colimits, and extensions.
- $\mathtt{Solid}$ is generated by compact projective objects of the form $\mathbb{Z}[S]^{\blacksquare}$.
- $\mathtt{Solid}$ is a reflective subcategory of $\mathtt{Cond}(\mathtt{Ab})$, i.e. the inclusion functor admits a left adjoint, given by the *solidification functor* $M \mapsto M^{\blacksquare}$.
- $\mathtt{Solid}$ is endowed with a unique symmetric monoidal structure, given by the *solid tensor product* $\otimes^{\blacksquare}$ for which the solidification functor is symmetric monoidal.
- $\mathcal{D}(\mathtt{Solid})$ inherits a derived solidification functor $C \mapsto C^{\mathrm{L}\blacksquare}$ and a derived solid tensor product $\otimes^{\mathrm{L}\blacksquare}$, which are the left derived functors of the solidification functor and the solid tensor product, respectively.

We now want to develop a similar theory for suitable condensed rings and their categories of modules. For this, we introduce the notion of **(pre-)analytic rings**.

Plan:
- define (pre-)analytic rings and consider examples
- discuss categorical properties, in particular, the symmetric monoidal structure
- discuss functoriality properties
- revisit the "non-archimedean nature" of the solid theory (i.e. what about $\mathbb{R}$)

A Sophisticated Definition

Recall: We defined the category of solid abelian groups in three steps:

- First, we declared what *free* solid abelian groups should look like,
- second, we defined solid abelian groups/complexes of solid abelian groups *relative* to our notion of free solid abelian groups, and
- third, we used a formal Categorical Lemma to obtain $\mathtt{Solid}$ as this *well-behaved subcategory* of $\mathtt{Cond}(\mathtt{Ab})$.

We now formalise and simultaneously generalise this procedure to allow, more generally, suitable categories of "condensed modules" instead of only condensed abelian groups. By this we mean the following.

Definition. Let R be a condensed ring. The category of ***R*-modules in condensed abelian groups** $\mathtt{Mod}_R^{\mathrm{cond}}$ has objects condensed abelian groups M such that $M(S)$ is endowed with the structure of an $R(S)$-module for each S, compatible with continuous maps, and morphisms $M \to N$ maps of condensed abelian groups such that $M(S) \to N(S)$ is $R(S)$-linear for each S. ⌟

Remark.
- If $R = \underline{\mathbb{Z}}$, then (unsurprisingly) $\mathtt{Mod}_{\underline{\mathbb{Z}}}^{\mathrm{cond}} = \mathtt{Cond}(\mathtt{Ab})$.
- A condensed ring R is a monoid object with respect to the tensor product $\otimes$ of condensed abelian groups and an R-module in condensed abelian groups is then a condensed abelian group M equipped with an action map $R \otimes M \to M$ (subject to certain coherence conditions).
- If R is commutative, then for R-modules M, N in condensed abelian groups, the coequaliser of the diagram

$$M \otimes R \otimes N \rightrightarrows M \otimes N$$

defines a tensor product $M \otimes_R N$ and hence a symmetric monoidal structure on $\mathtt{Mod}_R^{\mathrm{cond}}$.

- For a condensed ring R and R-modules M, N in condensed abelian groups, we define a complex of condensed abelian groups $\mathrm{R}\underline{\mathrm{Hom}}_R(M,N)(S) = \mathrm{RHom}_R(M \otimes \mathbb{Z}[\underline{S}], N)$. ⌟

Now, let us define an object, or rather a *framework*, combining the input of a choice of a condensed base ring and a class of "free complete" modules. Furthermore, we pin down a technical condition describing "completeness" in such a way, that, as output, this will produce a well-behaved subcategory of the category of modules in condensed abelian groups over our chosen condensed base ring.

Definition. A **pre-analytic ring $\boldsymbol{\mathcal{A}}$** consists of

(i) a condensed ring $\underline{\mathcal{A}}$,

(ii) a functor $\mathcal{A}[-]\colon \mathtt{Proj} \to \mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ taking finite disjoint unions to finite products, and

(iii) a natural transformation $\underline{-} \to \mathcal{A}[-]$.

We say that $\mathcal{A}$ is **analytic** if

(iv) for all complexes $C_\bullet$ in $\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$, with each C_i of the form $C_i = \bigoplus_T \mathcal{A}[T]$, the natural map

$$\mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C_\bullet) \to \mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[\underline{S}], C_\bullet)$$

is an isomorphism. ⌟

Remark. There is an ∞-categorical version of this definition, which is used in various secondary literature. Most of the results we will discuss below extend to this case (given the appropriate translations). ⌟

Let us try to explain the different moving pieces in more detail:

(i) $\underline{\mathcal{A}}$ is our base ring, determining which (algebraic) module structures we consider, that is, which category of modules in condensed abelian groups we consider.

(ii) $\mathcal{A}[-]\colon \mathtt{Proj} \to \mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ determines our choice of "free complete" $\underline{\mathcal{A}}$-modules in condensed abelian groups. We impose a sheaf condition to ensure that we can obtain from this a notion of "complete" $\underline{\mathcal{A}}$-modules.

(iii) For each S, the natural transformation yields a map $\underline{S} \to \mathcal{A}[S]$, which induces a "completion" map $\underline{\mathcal{A}}[\underline{S}] \to \mathcal{A}[S]$. We want to think of $\mathcal{A}[S]$ as the "completion" of the free $\underline{\mathcal{A}}$-module $\underline{\mathcal{A}}[\underline{S}]$.

(iv) We use $\underline{\mathcal{A}}[\underline{S}]$ and its "completion" $\mathcal{A}[S]$ to test for "completeness", as we did in our definition of solid abelian groups. Requiring the given isomorphisms pins down, in particular, $\mathcal{A}[S]$ as a genuine "complete" module.

We consider the following examples of pre-analytic rings.

a) The pre-analytic ring $\mathbb{Z}_\blacksquare$ is given by

  (i) the (discrete) condensed ring $\underline{\mathbb{Z}_\blacksquare} = \underline{\mathbb{Z}}$,

  (ii) the functor $\mathbb{Z}_\blacksquare[-]\colon \mathtt{Proj} \to \mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathbb{Z}}} = \mathtt{Cond(Ab)}$, $S \mapsto \mathbb{Z}[S]^\blacksquare := \varprojlim \mathbb{Z}[\underline{S_i}]$, and

  (iii) the natural transformation, defined from $\underline{S} = \underline{\varprojlim S_i} \to \varprojlim \mathbb{Z}[\underline{S_i}] = \mathbb{Z}_\blacksquare[S]$.

b) For a condensed ring R, the pre-analytic ring R is given by

  (i) the condensed ring $\underline{R} = R$,

  (ii) the functor $R[-]\colon \mathtt{Proj} \to \mathtt{Mod}^{\mathrm{cond}}_R$, $S \mapsto R[\underline{S}]$, and

  (iii) the natural transformation $\underline{S} \to R[S] = R[\underline{S}]$.

c) The pre-analytic ring $\mathbb{Z}_{p,\blacksquare}$ is given by

  (i) the condensed ring $\underline{\mathbb{Z}_{p,\blacksquare}} = \underline{\mathbb{Z}_p}$,

  (ii) the functor $\mathbb{Z}_{p,\blacksquare}[-]\colon \mathtt{Proj} \to \mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathbb{Z}_p}}$, $S \mapsto (\underline{\mathbb{Z}_p}[\underline{S}])^\blacksquare \overset{!}{=} \varprojlim \underline{\mathbb{Z}_p}[\underline{S_i}]$, and

  (iii) the natural transformation, defined from $\underline{S} \to \underline{\mathbb{Z}_p}[\underline{S}] \to \mathbb{Z}_{p,\blacksquare}[S]$.

d) For a discrete ring A, the pre-analytic ring $(A,\mathbb{Z})_\blacksquare$ is given by

(i) the (discrete) condensed ring $\underline{(A,\mathbb{Z})_\blacksquare} = \underline{A}$,

(ii) the functor $(A,\mathbb{Z})_\blacksquare[-]\colon \texttt{Proj} \to \texttt{Mod}^{\mathrm{cond}}_{\underline{A}}$, $S \mapsto \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \underline{A} \overset{!}{=} \mathbb{Z}_\blacksquare[S] \otimes \underline{A}$, and

(iii) the natural transformation $\underline{S} \to \mathbb{Z}_\blacksquare[S] \to \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \underline{A}$.

e) The pre-analytic ring $(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare$ is given by

(i) the condensed ring $\underline{(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare} = \underline{\mathbb{Q}_p}$,

(ii) the functor $(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare[-]\colon \texttt{Proj} \to \texttt{Mod}^{\mathrm{cond}}_{\underline{\mathbb{Q}_p}}$, $S \mapsto \mathbb{Z}_{p,\blacksquare}[S] \otimes^\blacksquare_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Q}_p} \overset{!}{=} \mathbb{Z}_{p,\blacksquare}[S] \otimes_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Q}_p}$, and

(iii) the natural transformation $\underline{S} \to \mathbb{Z}_{p,\blacksquare}[S] \to \mathbb{Z}_{p,\blacksquare}[S] \otimes^\blacksquare_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Q}_p}$.

f) For a finitely generated $\mathbb{Z}$-algebra A, the pre-analytic ring $A_\blacksquare$ is given by

(i) the (discrete) condensed ring $\underline{A_\blacksquare} = \underline{A}$,

(ii) the functor $A_\blacksquare[-]\colon \texttt{Proj} \to \texttt{Mod}^{\mathrm{cond}}_{\underline{A}}$, $S \mapsto A_\blacksquare[S] = \varprojlim \underline{A}[\underline{S_i}]$, and

(iii) the natural transformation, defined from $\underline{S} = \underline{\varprojlim S_i} \to \varprojlim \underline{A}[\underline{S_i}] = A_\blacksquare[S]$.

We can rephrase part of the assertions of the last talk as the observation that $\mathbb{Z}_\blacksquare$ is an analytic ring. Thus, the framework of analytic rings includes the notion of solid abelian groups. Using this, we can deduce the following.

Proposition. *The pre-analytic rings R, $\mathbb{Z}_{p,\blacksquare}$, $(A,\mathbb{Z})_\blacksquare$ and $(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare$ are analytic.*

Proof. Observe that for the pre-analytic ring R, $\underline{R} = R$ and therefore $R[S] = \underline{R}[\underline{S}]$. Since the natural comparison map $\underline{R}[\underline{S}] \to R[S]$, from which analyticity is defined, is an isomorphism (in fact, the identity), we conclude that R is analytic.

We can prove the remaining three cases simultaneously. Let $\mathcal{A}$ denote any of the pre-analytic rings given and abbreviate with $A = \underline{\mathcal{A}}$ the underlying condensed ring. Let $C_\bullet$ be a complex of A-modules in condensed abelian groups with each C_i of the form $C_i = \bigoplus_T \mathcal{A}[T]$. We have to show that for each extremally disconnected set S the natural map

$$\mathrm{R}\underline{\mathrm{Hom}}_A(\mathcal{A}[S], C_\bullet) \to \mathrm{R}\underline{\mathrm{Hom}}_A(A[S], C_\bullet)$$

is an isomorphism. We make the following claims:

(i) $\mathcal{A}[S] \cong \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare A \cong \mathbb{Z}_\blacksquare[S] \otimes^{\mathrm{L}\blacksquare} A$.

(ii) $C_\bullet$ is in $\mathcal{D}(\texttt{Solid})$.

Assuming these claims, we obtain the desired isomorphism as the composition of the following isomorphisms

$$\begin{aligned}
\mathrm{R}\underline{\mathrm{Hom}}_A(A[S], C_\bullet) &\cong \mathrm{R}\underline{\mathrm{Hom}}_A(\mathbb{Z}[S] \otimes^{\mathrm{L}} A, C_\bullet) \\
&\cong \mathrm{R}\underline{\mathrm{Hom}}(\mathbb{Z}[S], C_\bullet) \\
&\cong \mathrm{R}\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S], C_\bullet) \\
\mathrm{R}\underline{\mathrm{Hom}}_A(\mathcal{A}[S], C_\bullet) &\cong \mathrm{R}\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S] \otimes^{\mathrm{L}\blacksquare} A, C_\bullet)
\end{aligned}$$

making use of the (derived) tensor-hom adjunction twice.

For (i), we observe that $\mathbb{Z}_\blacksquare[S] \otimes^{\mathrm{L}\blacksquare} A \cong \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare A$. Indeed, since $\mathbb{Z}_\blacksquare[S]$ is a projective solid abelian group, the derived solid tensor product $\mathbb{Z}_\blacksquare[S] \otimes^{\mathrm{L}\blacksquare} A$ is concentrated in degree 0. Thus, it suffices to show that $\mathcal{A}[S] \cong \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare A$. For $(A,\mathbb{Z})_\blacksquare$ this holds by definition. Moreover, if we have obtained the isomorphism for the pre-analytic ring $\mathbb{Z}_{p,\blacksquare}$, we can deduce that

$$(\mathbb{Q}_p,\mathbb{Z}_p)_\blacksquare[S] = \mathbb{Z}_{p,\blacksquare}[S] \otimes^\blacksquare_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Q}_p} \cong (\mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \underline{\mathbb{Z}_p}) \otimes^\blacksquare_{\underline{\mathbb{Z}_p}} \underline{\mathbb{Q}_p} \cong \mathbb{Z}_\blacksquare[S] \otimes^\blacksquare \underline{\mathbb{Q}_p}\,.$$

Thus, it only remains to show that $\mathbb{Z}_{p,\blacksquare}[S] = (\underline{\mathbb{Z}_p}[\underline{S}])^{\blacksquare} \cong \mathbb{Z}_{\blacksquare}[S] \otimes^{\blacksquare} \underline{\mathbb{Z}_p}$. Since $\underline{\mathbb{Z}_p}$ is a limit of solid abelian groups, $\underline{\mathbb{Z}_p}^{\blacksquare} = \underline{\mathbb{Z}_p}$. As the solidification is (symmetric) monoidal for the tensor product, we find that

$$\mathbb{Z}_{\blacksquare}[S] \otimes^{\blacksquare} \underline{\mathbb{Z}_p} \cong \mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} \underline{\mathbb{Z}_p}^{\blacksquare} \cong (\mathbb{Z}[\underline{S}] \otimes \underline{\mathbb{Z}_p})^{\blacksquare} \cong (\underline{\mathbb{Z}_p}[\underline{S}])^{\blacksquare}.$$

This shows that $\mathbb{Z}_{\blacksquare}[S] \otimes^{\blacksquare} \underline{\mathbb{Z}_p}$ is the solidification of $\mathbb{Z}_p[\underline{S}]$.
For (ii), we observe that since $\underline{\mathbb{Z}_p}$, $\underline{\mathbb{Q}_p}$, and a discrete condensed ring $\underline{A}$ can all be realised as a combination of limits and colimits of the solid abelian group $\underline{\mathbb{Z}}$, they are all solid abelian groups themselves. The same remains true when applying $\mathbb{Z}_{\blacksquare}[S] \otimes^{\mathrm{L}\blacksquare} -$. This shows that $\mathcal{A}[S]$ is solid in our cases of interests. In particular, the complex $C_\bullet$ lies in $\mathcal{D}(\texttt{Solid})$ since its cohomologies are constructed from limits and colimits involving only maps between solid abelian groups of the form $\bigoplus_T \mathcal{A}[T]$. ☐

We will see f) for the special case $A = \mathbb{Z}[T]$ in the next talk. Moreover, there is an analogue of d) and e) for general Huber pairs (A, A^+). Later we will see this for A discrete.

Categories of $\mathcal{A}$-modules

We now study the categories of modules in condensed abelian groups we obtain from analytic rings.

Theorem. *Let $\mathcal{A}$ be an analytic ring.*

(i) The full subcategory

$$\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}} \subset \texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$$

of all $\mathcal{A}$-modules M in condensed abelian groups such that for all extremally disconnected sets S, the map

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M) \to \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], M)$$

is an isomorphism, is an abelian subcategory closed under all limits, colimits, and extensions. The objects $\mathcal{A}[S]$ for S extremally disconnected form a family of compact projective generators. The inclusion admits a left adjoint

$$\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} \to \texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}, \qquad M \mapsto \mathcal{A} \otimes_{\underline{\mathcal{A}}} M$$

that is the unique colimit-preserving extension of $\underline{\mathcal{A}}[S] \to \mathcal{A}[S]$. If $\underline{\mathcal{A}}$ is commutative, there is a unique symmetric monoidal tensor product $\otimes_{\mathcal{A}}$ on $\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ making the functor

$$\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} \to \texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}, \qquad M \mapsto \mathcal{A} \otimes_{\underline{\mathcal{A}}} M$$

symmetric monoidal.

(ii) The functor

$$\mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}) \to \mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}})$$

is fully faithful, and its essential image is closed under all limits and colimits given by those $C_\bullet \in \mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}})$ such that for all extremally disconnected S, the map

$$\mathrm{RHom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C_\bullet) \to \mathrm{RHom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C_\bullet)$$

is an isomorphism; in that case, also the $\mathrm{R}\underline{\mathrm{Hom}}$'s agree. An object $C_\bullet \in \mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}})$ lies in $\mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ if and only if $\mathrm{H}^i(C_\bullet)$ are in $\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$. The inclusion $\mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}) \subset \mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}})$ admits a left-adjoint

$$\mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}) \to \mathcal{D}(\texttt{Mod}^{\mathrm{cond}}_{\mathcal{A}}), \qquad C_\bullet \mapsto \mathcal{A} \otimes^{\mathrm{L}}_{\underline{\mathcal{A}}} C_\bullet$$

that is the left derived functor of $M \mapsto M \otimes_{\underline{\mathcal{A}}} \mathcal{A}$. If $\underline{\mathcal{A}}$ is commutative, there is a unique symmetric monoidal tensor *product* $\otimes^{\mathrm{L}}_{\mathcal{A}}$ *on* $\mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ *making the functor*

$$\mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}) \to \mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}), \qquad C_\bullet \mapsto \mathcal{A} \otimes^{\mathrm{L}}_{\underline{\mathcal{A}}} C_\bullet$$

symmetric monoidal.

Remark. Contrary to our notation, in general cases it could happen that $\otimes^{\mathrm{L}}_{\mathcal{A}}$ is *not* the left derived functor of $\otimes_{\mathcal{A}}$.⌟

Proof. Almost all assertions follow immediately from the Categorical Lemma we established while developing the theory of solid abelian groups. This lemma applies, since the notion of a pre-analytic ring yields the correct setup, while the analyticity condition translates precisely into the necessary assumption.

It remains to prove that the R$\underline{\mathrm{Hom}}$'s agree and to equip $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ as well as $\mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ with a symmetric monoidal structure. This can be done as follows:

- Let $C_\bullet$ be an object of $\mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}})$. If $C_\bullet$ is bounded from below, then the isomorphism is precisely given as requirement that $\mathcal{A}$ is analytic. The general case follows from the type of limit argument we have seen in the proof of the Categorical Lemma (keyword: Postnikov limit).
- If $\underline{\mathcal{A}}$ is commutative, we can define $M \otimes_{\underline{\mathcal{A}}} N$ as the coequaliser of the diagram

$$M \otimes \underline{\mathcal{A}} \otimes N \rightrightarrows M \otimes N$$

  with $\underline{\mathcal{A}}$-modules M, N in condensed abelian groups. This gives the a symmetric monoidal structure $\otimes_{\underline{\mathcal{A}}}$ on $\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$ and by taking left derived functors this induces a symmetric monoidal structure $\otimes^{\mathrm{L}}_{\underline{\mathcal{A}}}$ on $\mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}})$.
- If $\underline{\mathcal{A}}$ is commutative, we can define $\otimes_{\mathcal{A}}$ as

$$M \otimes_{\mathcal{A}} N = \mathcal{A} \otimes_{\underline{\mathcal{A}}} (M \otimes_{\underline{\mathcal{A}}} N)$$

  in analogy to the case of the solid tensor product of solid abelian groups. It requires some additional work to show that this descends to the derived category. In particular, we cannot simply take the left derived functor of $\otimes_{\mathcal{A}}$ since it is in general not true that for extremally disconnected sets S, T the $\underline{\mathcal{A}}$-module $\mathcal{A}[S \times T]$ is concentrated in degree 0 (keyword: $\otimes$-ideal). □

Remark.

- For an analytic ring $\mathcal{A}$ we let $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}})$ denote the **derived category of $\mathcal{A}$-modules**.
- For consistency, we will redefine our notation for solid abelian groups to match our new point of view using the analytic ring $\mathbb{Z}_\blacksquare$. Thus, we will write

$$\mathtt{Mod}^{\mathrm{cond}}_{\mathbb{Z}_\blacksquare} = \mathtt{Solid}$$

  for the category of solid abelian groups, we will denote by

$$\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathbb{Z}}} \to \mathtt{Mod}^{\mathrm{cond}}_{\mathbb{Z}_\blacksquare}, \qquad M \mapsto \mathbb{Z}_\blacksquare \otimes_{\underline{\mathbb{Z}}} M$$

  the solidification functor, and the symmetric monoidal tensor product $\otimes^\blacksquare$ is now denoted by $\otimes_{\mathbb{Z}_\blacksquare}$. Similarly, the derived solidification functor will be denoted by

$$\mathcal{D}(\mathtt{Cond}(\mathtt{Ab})) \to \mathcal{D}(\mathbb{Z}_\blacksquare), \qquad C_\bullet \mapsto \mathbb{Z}_\blacksquare \otimes^{\mathrm{L}}_{\underline{\mathbb{Z}}} C_\bullet$$

  and the derived symmetric monoidal tensor product $\otimes^{\mathrm{L}\blacksquare}$ is now denoted by $\otimes^{\mathrm{L}}_{\mathbb{Z}_\blacksquare}$. ⌟

Having defined our categories of interest, we turn to questions of functoriality. For this, we first define what types of morphisms of pre-analytic and analytic rings we consider

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be pre-analytic rings. A **map of pre-analytic rings** $\mathcal{A} \to \mathcal{B}$ is a map of condensed rings $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ together with a natural transformation $\mathcal{A}[-] \to \mathcal{B}[-]$ commuting with the natural transformations from $\underline{-}$ and such that for each extremally disconnected set S the map $\mathcal{A}[S] \to \mathcal{B}[S]$ is $\underline{\mathcal{A}}$-linear. ⌟

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be analytic rings. A **map of analytic rings** $\mathcal{A} \to \mathcal{B}$ is a map of condensed rings $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ such that for each extremally disconnected set S the $\underline{\mathcal{A}}$-module $\mathcal{B}[S]$ lies in $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$. ⌟

Consider a map of analytic rings $\mathcal{A} \to \mathcal{B}$ and let S be an extremally disconnected set. As the $\underline{\mathcal{A}}$-module $\mathcal{B}[S]$ is contained in $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$, we find that

$$\mathrm{Hom}(\underline{S}, \mathcal{B}[S]) \cong \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], \mathcal{B}[S]) \cong \mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], \mathcal{B}[S]) \,.$$

Hence the natural map $S \to \mathcal{B}[S]$ induces, functorially, an $\underline{\mathcal{A}}$-linear map $\mathcal{A}[S] \to \mathcal{B}[S]$ which, by construction, commutes with the natural transformations from $\underline{-}$. In particular, a map of analytic rings induces a map of the underlying pre-analytic rings.

The converse holds under a mild technical assumption.

Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be analytic rings. Let $\mathcal{A} \to \mathcal{B}$ by a map of the underlying pre-analytic rings. Assume that for all extremally disconnected sets S with a map $\underline{S} \to \underline{\mathcal{A}}$, inducing a unique map $\mathcal{A}[S] \to \mathcal{A}[*]$ in $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$, and, from the composite $\underline{S} \to \underline{\mathcal{A}} \to \underline{\mathcal{B}}$, a unique map $\mathcal{B}[S] \to \mathcal{B}[*]$ in $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{B}}$, the diagram*

$$\begin{array}{ccc} \mathcal{A}[S] & \longrightarrow & \mathcal{A}[*] \\ \downarrow & & \downarrow \\ \mathcal{B}[S] & \longrightarrow & \mathcal{B}[*] \end{array}$$

commutes. Then $\mathcal{A} \to \mathcal{B}$ is a map of analytic rings.

Proof. Omitted (Appendix to Lecture VII). □

Our definition of maps of analytic rings is chosen such that the following functoriality properties become true.

Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be analytic rings and let $\mathcal{A} \to \mathcal{B}$ be a map of analytic rings.*

(i) The composite functor

$$\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} \xrightarrow{\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} -} \mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{B}}} \xrightarrow{\mathcal{B} \otimes_{\underline{\mathcal{B}}} -} \mathtt{Mod}^{\mathrm{cond}}_{\mathcal{B}}$$

factors over $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$, via a functor denoted

$$\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}} \to \mathtt{Mod}^{\mathrm{cond}}_{\mathcal{B}}, \qquad M \mapsto \mathcal{B} \otimes_{\mathcal{A}} M \,.$$

(ii) The composite functor

$$\mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}) \xrightarrow{\underline{\mathcal{B}} \otimes^{\mathrm{L}}_{\underline{\mathcal{A}}} -} \mathcal{D}(\mathtt{Mod}^{\mathrm{cond}}_{\underline{\mathcal{B}}}) \xrightarrow{\mathcal{B} \otimes^{\mathrm{L}}_{\underline{\mathcal{B}}} -} \mathcal{D}(\mathcal{B})$$

factors over $\mathcal{D}(\mathcal{A})$, via a the left derived functor

$$\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}), \qquad C_\bullet \mapsto \mathcal{B} \otimes^{\mathrm{L}}_{\mathcal{A}} C_\bullet$$

of $M \mapsto M \otimes_{\mathcal{A}} \mathcal{B}$.

Proof. Observe here that

$$\mathcal{B} \otimes_{\underline{\mathcal{B}}} (\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{A}}[S]) = \mathcal{B} \otimes_{\underline{\mathcal{B}}} \underline{\mathcal{B}}[S] = \mathcal{B}[S]$$

which, by assumption, lies in $\mathtt{Mod}^{\mathrm{cond}}_{\mathcal{A}}$. As these generate, we conclude (i). Then (ii) follows from (i) as $C_\bullet \in \mathcal{D}(\mathcal{A})$ if and only if $\mathrm{H}^i(C_\bullet) \in \mathcal{D}(\mathcal{A})$. We finally conclude that the functor in (ii) is the left derived functor of $M \mapsto \mathcal{B} \otimes_{\mathcal{A}} M$ since it commutes with colimits and maps $\mathcal{A}[S]$ to $\mathcal{B}[S]$. □

Archimedean Theory

The theory of solid abelian groups is non-archimedean in the sense that $\underline{\mathbb{R}}^{\blacksquare} = \mathbb{Z}_{\blacksquare} \otimes \underline{\mathbb{R}} = 0$. Thus, a naive definition of pre-analytic ring $\mathbb{R}_{\blacksquare}$ with underlying ring $\underline{\mathbb{R}}$ and "free complete" modules $\mathbb{R}_{\blacksquare}[S] = (\underline{\mathbb{R}}[\underline{S}])^{\blacksquare}$, as we do in the non-archimedean case, cannot work. Indeed,

$$\mathbb{R}_{\blacksquare}[S] = (\underline{\mathbb{R}}[\underline{S}])^{\blacksquare} = (\underline{\mathbb{Z}}[\underline{S}] \otimes \underline{\mathbb{R}})^{\blacksquare} = \underline{\mathbb{Z}}[\underline{S}]^{\blacksquare} \otimes^{\blacksquare} \underline{\mathbb{R}}^{\blacksquare} = \mathbb{Z}_{\blacksquare}[S] \otimes^{\blacksquare} \underline{\mathbb{R}} = 0$$

since $\underline{\mathbb{R}}^{\blacksquare} = 0$. Thus, $\mathbb{R}_{\blacksquare}[S] = 0$.

To avoid this problem, we could try to make use of the measure-theoretic perspective, which already part of our development of the theory of solid abelian groups. More precisely, recall that

$$\mathbb{Z}_{\blacksquare}[S] = \underline{\mathrm{Hom}}(\underline{\mathrm{C}(S,\mathbb{Z})}, \underline{\mathbb{Z}}) = \underline{\mathcal{M}(S,\mathbb{Z})}$$

is the space of $\mathbb{Z}$-valued measures on S. There is a similar interpretation for some of the other (pre-)analytic rings we have considered so far. More precisely, we find that

$$\mathbb{Z}_{p,\blacksquare}[S] = \underline{\mathcal{M}(S,\mathbb{Z}_p)}\,, \quad A_{\blacksquare}[S] = \underline{\mathcal{M}(S,A)}\,, \quad \text{and} \quad (\mathbb{Q}_p,\mathbb{Z}_p)_{\blacksquare}[S] = \underline{\mathcal{M}^b(S,\mathbb{Q}_p)}\,,$$

where A is a finitely generated $\mathbb{Z}$-algebra and $\mathcal{M}^b(S,\mathbb{Q}_p)$ is the space of *bounded* $\mathbb{Q}_p$-valued measures on S. Thus, a reasonable candidate for the "free complete" $\underline{\mathbb{R}}$-modules in condensed abelian groups could be

$$\mathbb{R}_{\ell^1}[S] = \underline{\mathcal{M}^b(S,\mathbb{R})}\,,$$

that is, the space of bounded $\mathbb{R}$-valued measures on S (with the topology dual to that of the Banach space $\mathrm{C}(S,\mathbb{R})$). This does not yield an analytic ring either. However, this time this is due to more subtle reasons, exemplified by the existence of *Ribe extensions* implying the following.

Proposition. *For any infinite extremally disconnected set S, one has*

$$\mathrm{Ext}^1_{\underline{\mathbb{R}}}(\mathbb{R}_{\ell^1}[S], \mathbb{R}_{\ell^1}[S]) \neq 0\,.$$

This is not possible for an analytic ring. Indeed, suppose that $\mathcal{A}$ is analytic. Then we know that for all extremally disconnected S the $\mathcal{A}$-modules $\mathcal{A}[S]$ is projective. Thus, for $i > 0$ we conclude that

$$\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{A}[S], \mathcal{A}[S]) = 0\,.$$

The situation can be remedied, but this leads to the more involved notion of **liquid $\mathbb{R}$-vector spaces**.